Strategies: A logic - automata study
Lecture 3: Game logic and its descendants

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- Existence of winning strategies used to analyze possibility of automaton offering suitable response for all possible provocations from an uncertain environment.
- Important applications in control synthesis and thus in the design and verification of systems.
- An emphasis on the size of *memory* needed.

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- Logical notions like satisfiability are reduced to existence of winning strategies for one of the players.
- A tradition of use in model theory, for model construction, comparing models etc.

Economists' games

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- Study equilibria to predict rational play; an emphasis on quantitative solutions.
- A tradition of use in market design, information economics and some political analysis.

Structure of strategies

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- Certainly, this is true of the studies above.

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- This suggests compositional structure in strategies.
- Something that logic is particularly good at.
- And this is best done by looking for compositional structure in games. For this viewing games as programs is useful.

Structured programs

Let Σ be a set of *atomic* programs. More complex programs are built from these using program composition.

$$Pr ::= a \in \Sigma \mid \pi_1 + \pi_2 \mid \pi_1 ; \pi_2 \mid \pi^*$$

- This is the class of *regular* or finite-state programs.

Regular operations

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- Assignment statements, input / output are all among atomic programs in Σ.

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- Alternatively, if we were given a set of states X, we can consider the program to be a mechanism that "achieves" X when started from s.
- If we were to see X as a 'goal', a program then sounds very much like a strategy to achieve the goal.

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If α is a property (defined on states), we can think of a program π as a set of pairs (α, β); if π is started in a state where α holds, when π terminates, the resulting state is one in which β holds.

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- This is Propositional dynamic logic.

Nondeterministic programs

Consider the following (non-deterministic) program: let x, y be natural numbers.
(**if** $x \geq y$ **then** $y := y + 1$) or (**if** $x \leq y$ **then** $x := x + 1$)

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- Such programs can be thought as *1-player games*: when $x = y$, the player (Nature) has a choice of which transition to do.

Programs as games

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$\pi(X) = Y$ denotes the following:

> *Y is the set of states starting from which Nature has a strategy to reach the set X of states.*

Two player games

Does the addition of a second player make any essential difference ?

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- This idea leads us to Propositional game logic, which is similar to program logic, but admitting a player and an opponent.

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- $(a + b); (e + f)^d$ stands for the game where one player chooses between a and b and then the other player chooses between e and f.
- The formulas of the logic are as defined before: $< \pi > \alpha$ now denotes that Player I has a strategy for playing game π and achieving α.

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- They go down recursively into the formula: I gets a turn when the connective is $\vee$ or $\exists$. Similarly, II moves for $\wedge$ and $\forall$. Negation is role reversal.
- These are the game logic operators. Fixed point constructors in the logic are associated with game iteration.

Game logic [Parikh 83]

A logic to reason about determined two person zero sum games.

Syntax

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- $\Phi := p \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \langle\gamma\rangle\alpha$.
- $\Gamma := g \in \Gamma_0 \mid \gamma_1 ; \gamma_2 \mid \gamma_1 \cup \gamma_2 \mid \gamma^* \mid \gamma^d$.
  - Sequential composition - $\gamma_1 ; \gamma_2$.
  - Choice - $\gamma_1 \cup \gamma_2$.
  - Iteration - γ^*.
  - Dual - γ^d.

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Interpretation for games

- Final outcomes which players can enforce.

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Interpretation for games

- Final outcomes which players can enforce.
- Set of states S.
- Effectivity relation - $E_g \subseteq S \times 2^S$
  - $(s, X) \in E_g$ iff starting at s, in game g, player 1 can enforce the outcome to be in X.

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- $\Phi := p \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \langle\gamma\rangle\alpha$.
- $\Gamma := g \in \Gamma_0 \mid \gamma_1;\gamma_2 \mid \gamma_1 \cup \gamma_2 \mid \gamma^d \mid \gamma^*$.

Model $M = (S, \{E_g \mid g \in \Gamma_0\}, V)$.

Neighbourhood semantics

- $M, s \models \langle\gamma\rangle\alpha$ iff $\exists(s, X) \in E_\gamma$ such that $X \subseteq \{s' \mid M, s' \models \alpha\}$.

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Neighbourhood semantics

- $M, s \models \langle\gamma\rangle\alpha$ iff $\exists(s, X) \in E_\gamma$ such that $X \subseteq \{s' \mid M, s' \models \alpha\}$.
- There exists a strategy in game γ to ensure α.

Determined games

We talk about two person zero sum games of perfect information in this logic.

- The two players cannot have winning strategies for complementary winning positions, thus $\neg(<\pi>\alpha \wedge <\pi^{d}>\neg\alpha)$ is a valid formula, for every game π.

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- The two players cannot have winning strategies for complementary winning positions, thus $\neg(<\pi>\alpha\wedge<\pi^{d}>\neg\alpha)$ is a valid formula, for every game π.
- All games are determined, and hence $(<\pi>\alpha\vee<\pi^{d}>\neg\alpha)$ is valid.

Axiom system

- All the substitutional instances of tautologies of PC.
- $\langle g_1 \cup g_2 \rangle \alpha \equiv \langle g_1 \rangle \alpha \vee \langle g_2 \rangle \alpha$.
- $\langle g_1 ; g_2 \rangle \alpha \equiv \langle g_1 \rangle \langle g_2 \rangle \alpha$.
- $\langle g^* \rangle \alpha \equiv \alpha \vee \langle g \rangle \langle g^* \rangle \alpha$.
- $\langle g^d \rangle \alpha \equiv \neg \langle g \rangle \neg \alpha$.

Inference rules

$$(MP)\ \frac{\alpha, \quad \alpha \supset \beta}{\beta} \qquad (NG)\ \frac{\alpha \supset \beta}{\langle g \rangle \alpha \supset \langle g \rangle \beta}$$

$$(IND)\ \frac{\langle g \rangle \alpha \supset \alpha}{\langle g^* \rangle \alpha \supset \alpha}$$

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- The system without the duality axiom is easily seen to be complete for the dual-free fragment of the logic.
- In 1983, Parikh conjectured that the system presented is indeed complete for game logic.
- This remains an interesting open problem.

Technical results

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- This is the same as that for propositional dynamic logic.
- Model checking game logic is equivalent to the same problem for the modal μ-calculus.
- Complexity of model checking is in $NP \cap co - NP$: a major open problem asks if it is in P.

Remarks on game logic

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- Game expressions define rules for constructing extensive game forms over internal positions, while atomic game forms and utilities are provided in terms of world states.
- This mix of game theoretical concepts and computer science leads to a non-classical notion of great potential.

Relation to program logics

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Relation to program logics

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- We can define a notion of *bisimulation* over game models and show that formulas of game logic are invariant under bisimulation.
- When we consider the natural notion of *modal alternation depth* of a formula, we find that the alternation hierarchy for game logic formulas does not collapse.
- There is a translation of game logic into propositional μ-calculus with 2 variables (and 1 variable does not suffice).

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- Three mappings ^{X}, ^{Y} and ^{F} are defined.
- ^{X}, ^{Y} translate game expressions into formulas with one variable X and Y respectively.
- ^{F} is the propositional translation.

The map ^{X}

- $g^X = \Diamond X$.
- $(g_1 + g_2)^X = g_1^X \vee g_2^X$.
- $(g_1; g_2)^X = g_1^X[X := g_2^X]$.
- $(g^d)^X = \neg g^X[X := \neg X]$.
- $(g^*)^X = \mu Y.(X \vee g^Y)$.

The map ^{Y}

- $g^Y = \Diamond Y$.
- $(g_1 + g_2)^Y = g_1^Y \vee g_2^Y$.
- $(g_1; g_2)^Y = g_1^Y[Y := g_2^Y]$.
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The map ^{F}

- $p^F = p$.
- $(\neg\phi)^F = \neg\phi^F$.
- $(\phi_1 \vee \phi_2)^F = \phi_1^F \vee \phi_2^F$.
- $(< g > \phi)^F = g^X[X := \phi^F]$.

The map ^{F}

- $p^F = p$.
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- $(\phi_1 \vee \phi_2)^F = \phi_1^F \vee \phi_2^F$.
- $(< g > \phi)^F = g^X[X := \phi^F]$.

We can then show that ϕ and ϕ^F are equivalent.

Departures

Game logic provides a point of departure for the study of logic and games.

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- Multi-player games.
- Overlapping objectives.
- More interesting game operators.

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At the logical level:

- Game composition and structured strategies are *not* independent entities.
- Games and strategies need to be composed together.

N-person game logic

Game logic: $\langle\gamma\rangle\alpha$

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N-person game logic: $\langle\gamma, i\rangle\alpha$

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- $\Phi := p \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \langle\gamma, i\rangle\alpha$.
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Semantics

$\mathcal{M} = (S, \{\rho_g^i \mid g \in \Gamma_0\}, V)$, where $\rho_g^i \subseteq S \times \mathcal{P}(S)$ is monotonic.

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- How do we define the semantics?
- What is the truth definition of $\langle \gamma_1 \cup \gamma_2, i \rangle \varphi$?
- What kind of validities does this logic have?
- Is $\neg(\langle \gamma, i \rangle \varphi \wedge \langle \gamma, j \rangle \neg \varphi)$ valid?

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- We consider two ways of looking at it : Intersection and interleaving
- Intersection: neighbourhood models
- Interleaving: tree models

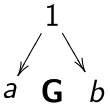
Simultaneous games

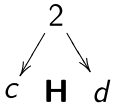
- Game models: $\mathcal{M} = (S, \{\rho_g^i \mid g \in \Gamma\}, V)$, where $\rho_g^i \subseteq S \times \mathcal{P}(S)$ is monotonic.
- Simultaneous game models: $\mathcal{M} = (S, \{\rho_g^i \mid g \in \Gamma\}, V)$, where $\rho_g^i \subseteq S \times \mathcal{P}(\mathcal{P}(S))$ is monotonic.
  The set lifting is considered to differentiate between union, intersection and parallel composition of games.

A toy example

$$\begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ a & \mathbf{G} & b \end{tikzcd} \qquad \begin{tikzcd} & 2 \arrow[dl] \arrow[dr] & \\ c & \mathbf{H} & d \end{tikzcd}$$

A toy example





1's ability : $\{\{a\}\}, \{\{b\}\}$.
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1's ability : $\{\{c\}, \{d\}\}$.
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A toy example

1's ability : {{a, c}, {a, d}}, {{b, c}, {b, d}}.
2's ability : {{a, c}, {b, c}}, {{a, d}, {b, d}}.

Each outcome state is a set read ‘conjunctively’, but players have choices leading to sets of these read ‘disjunctively’ as in case of game logic.

Simultaneous two-player game logic

Language:
$\gamma := g \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^d \mid \gamma \times \gamma$
$\phi := \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid \langle \gamma, i \rangle \phi$

Simultaneous game model: $\mathcal{M} = (S, \{\rho^i_g \mid g \in \Gamma\}, V)$

$\mathcal{M}, s \models \langle \gamma, i \rangle \phi$ iff there exists $X : s\rho^i_\gamma X$ and $\forall x \in \cup X : \mathcal{M}, x \models \varphi$

Composite games

$$s\rho^1_{G\cup G'}X \quad \text{iff} \quad s\rho^1_G X \text{ or } s\rho^1_{G'}X.$$

$$s\rho^2_{G\cup G'}X \quad \text{iff} \quad s\rho^2_G X \text{ and } s\rho^2_{G'}X.$$

$$s\rho^1_{G^d}X \quad \text{iff} \quad s\rho^2_G X.$$

$$s\rho^2_{G^d}X \quad \text{iff} \quad s\rho^1_G X.$$

$$s\rho^i_{G;G'}X \quad \text{iff} \quad \exists U : s\rho^i_G U \text{ and for each } u \in \bigcup U,\ u\rho^i_{G'}X.$$

$$s\rho^i_{G\times G'}X \quad \text{iff} \quad \exists T, \exists W : s\rho^i_G T \text{ and } s\rho^i_{G'}W \text{ and } X = \{t \cup w : t \in T \text{ and } w \in W\}.$$

Complete axiomatization

a) all propositional tautologies and inference rules

b) if $\vdash \phi \rightarrow \psi$ then $\vdash \langle \gamma, i \rangle \phi \rightarrow \langle \gamma, i \rangle \psi$

c) reduction axioms:

$$\langle \alpha \cup \beta, 1 \rangle \phi \leftrightarrow \langle \alpha, 1 \rangle \phi \vee \langle \beta, 1 \rangle \phi$$
$$\langle \alpha \cup \beta, 2 \rangle \phi \leftrightarrow \langle \alpha, 2 \rangle \phi \wedge \langle \beta, 2 \rangle \phi$$
$$\langle \gamma^d, 1 \rangle \phi \leftrightarrow \langle \gamma, 2 \rangle \phi$$
$$\langle \gamma^d, 2 \rangle \phi \leftrightarrow \langle \gamma, 1 \rangle \phi$$
$$\langle \alpha; \beta, i \rangle \phi \leftrightarrow \langle \alpha, i \rangle \langle \beta, i \rangle \phi$$
$$\langle \alpha \times \beta, i \rangle \phi \leftrightarrow \langle \alpha, i \rangle \phi \wedge \langle \beta, i \rangle \phi$$

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$$\langle \alpha \times \beta, i \rangle \phi \leftrightarrow \langle \alpha, i \rangle \phi \wedge \langle \beta, i \rangle \phi$$

Consider $\langle \alpha \times \alpha, i \rangle \phi$!

Complete axiomatization

a) all propositional tautologies and inference rules

b) if $\vdash \phi \to \psi$ then $\vdash \langle \gamma, i \rangle \phi \to \langle \gamma, i \rangle \psi$

c) reduction axioms:

$$
\begin{aligned}
&\langle \alpha \cup \beta, 1 \rangle \phi \leftrightarrow \langle \alpha, 1 \rangle \phi \vee \langle \beta, 1 \rangle \phi \\
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&\langle \alpha; \beta, i \rangle \phi \leftrightarrow \langle \alpha, i \rangle \langle \beta, i \rangle \phi \\
&\langle \alpha \times \beta, i \rangle \phi \leftrightarrow \langle \alpha, i \rangle \phi \wedge \langle \beta, i \rangle \phi
\end{aligned}
$$

Consider $\langle \alpha \times \alpha, i \rangle \phi$! Composing the same games will give us back a single game, which do not agree with the spirit of what we were talking about earlier: Having games played in parallel, and copying strategies from one to the other.

Interleaving games

The general focus is on:

- an interleaving operator in the context of extensive form game trees looking into the structure of the games;
- information transfer from one game to another made possible by some common player enacting in all the games concerned;
- strategizing based on such information transfer.

This attempt

A formal study on the algebra of game composition: choice, iteration, sequential and parallel composition.

- A dynamic logic for game expressions extended with parallel composition.
- Extensive form games embedded in Kripke structures.
- Focus on interleavings of moves of players in the tree structure.

Composing game trees

Atomic games: Extensive form games

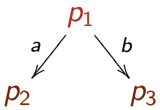
- Finite tree - nodes represent game positions labelled with players.
- Edge relation - specifies the moves which are enabled at a particular position.

Composite games:

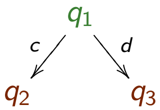
- choice $(g_1 \cup g_2)$;
- sequential composition $(g_1; g_2)$;
- iteration (g^*);
- parallel composition $(g_1 \| g_2)$.

Sequential composition

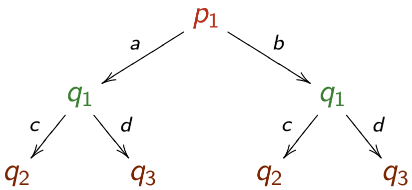
Sequential composition



T_1



T_2



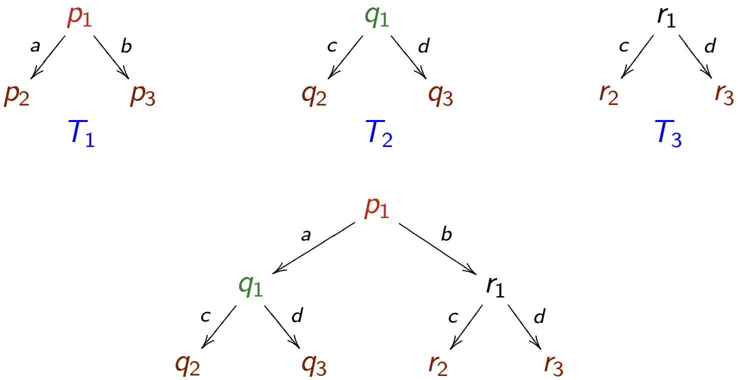
$T_1; T_2$

Sequential composition

$$
\begin{tikzcd}[column sep=small]
 & p_1 \arrow[dl, "a"'] \arrow[dr, "b"] & \\
p_2 & & p_3
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & q_1 \arrow[dl, "c"'] \arrow[dr, "d"] & \\
q_2 & & q_3
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & r_1 \arrow[dl, "c"'] \arrow[dr, "d"] & \\
r_2 & & r_3
\end{tikzcd}
$$

T_1 $\qquad$ T_2 $\qquad$ T_3

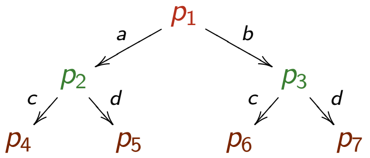
Sequential composition



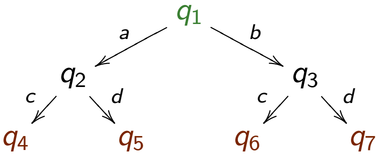
$$T \in T_1; \{T_2, T_3\}$$

Parallel composition

Same game; different players.

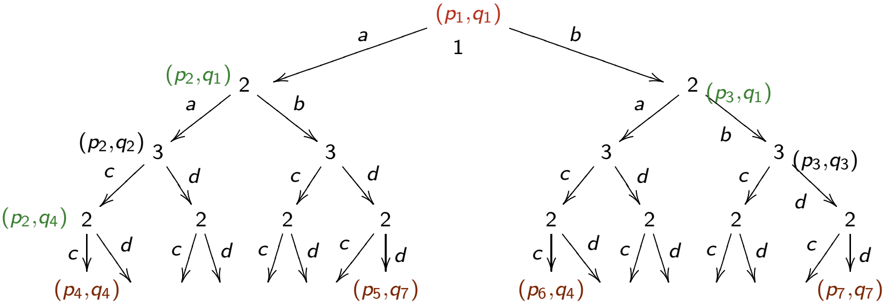


T_4



T_5

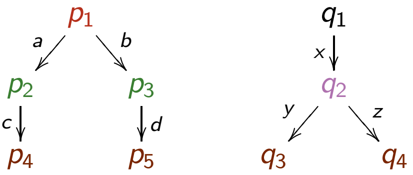
Parallel composition



$T \in T_4 \| T_5$

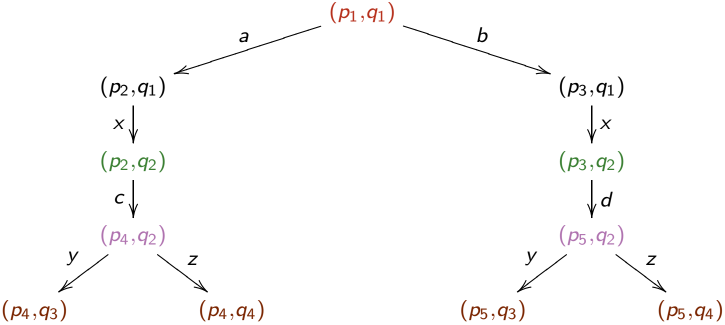
Parallel composition

Different games; different players.



T_6 T_7

Parallel composition



$$T \in T_6 \| T_7$$

The logic

Syntax

- $\Phi := p \in P \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \langle g, i \rangle \alpha$.
- $\Gamma := h \in \mathbb{H} \mid g_1 ; g_2 \mid g_1 \cup g_2 \mid g_1 \| g_2$.

The logic

Syntax

- $\Phi := p \in P \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \langle g, i \rangle \alpha$.
- $\Gamma := h \in \mathbb{H} \mid g_1 ; g_2 \mid g_1 \cup g_2 \mid g_1 \| g_2$.

Tree semantics

- $M, w \models \langle g, i \rangle \alpha$ iff there exists $X \in R_g^i$ such that X constitutes a valid tree, $root(X) = w$ and for all $v \in frontier(X)$, $M, v \models \alpha$. .

At a state w, a tree X in the "tree language" associated with g is enabled at w, and player i has a strategy (subtree) in it to ensure α.

Tree semantics

To define the semantic relation, we need to fix:

- Tree language associated with the game g.

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Atomic games: Extensive form games

- Finite tree - nodes represent game positions labelled with players.
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Model

- Model - Kripke structure.
  - A finite set of states W.
  - Labelled edge relation $\longrightarrow \subseteq W \times \Sigma \times W$.
  - Valuation function $V : W \to 2^P$.
  - Player labelling function $\lambda : W \to N$.

Model

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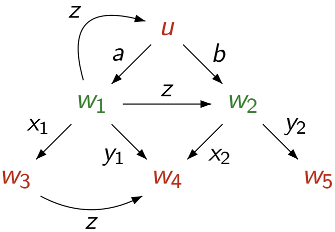
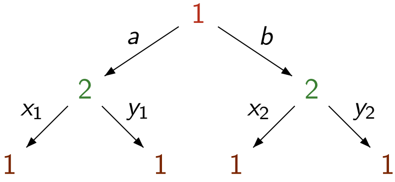
  - A finite set of states W.
  - Labelled edge relation $\longrightarrow \subseteq W \times \Sigma \times W$.
  - Valuation function $V : W \rightarrow 2^P$.
  - Player labelling function $\lambda : W \rightarrow N$.
- Tree language for atomic games h.
  - if h is enabled at w, $R_h^i(w) \subseteq 2^{(W \times W)^*}$ encodes the set of all available strategies for player i in the game h enabled at w.
  - $R_h^i = \bigcup_{w \in W} R_h^i(w)$.

Game enabled

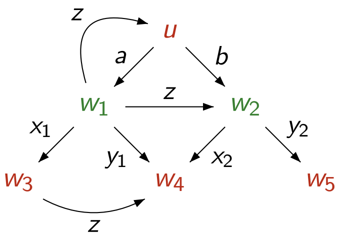


Model - M

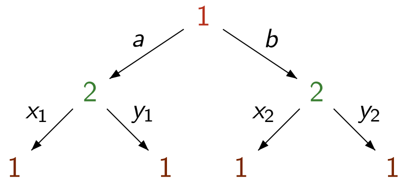


Game - h

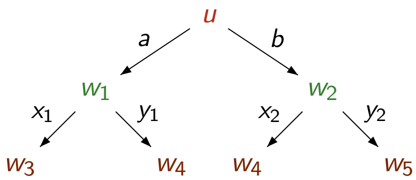
Game enabled



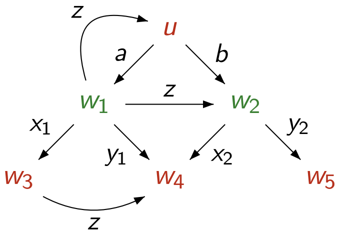
Model - M



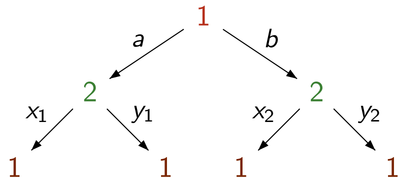
Game - h



Game enabled

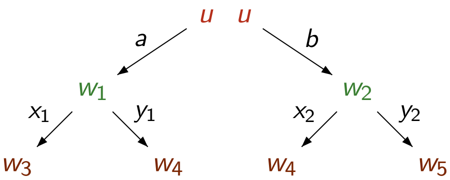


Model - M



Game - h

$$\{(u, w_1)(w_1, w_3), (u, w_1)(w_1, w_4)\},$$
$$\{(u, w_2)(w_2, w_4), (u, w_2)(w_2, w_5)\} \in R_h^1$$



Valid tree

Legal sequence ϱ: $(u, w)(v, x)$ in ϱ implies $w = v$.

$X \in 2^{(W \times W)^*}$ is a valid tree if:

- for all sequence $\varrho \in X$, ϱ is legal;
- X is prefix closed.

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Root and frontier

In a valid tree X,

- for all $\varrho, \varrho' \in X$, $\mathit{first}(\varrho)[1] = \mathit{first}(\varrho')[1] = \mathit{root}(X)$;
- $\mathit{frontier}(X) = \{\mathit{last}(\varrho)[2] \mid \varrho \in X\}$.

Composite game relations

- $X \in R^i_h \Longrightarrow$ tree structure $\mathfrak{T}(X)$
- finite game tree $\mathfrak{T} \Longrightarrow \mathfrak{f}(\mathfrak{T}) \in 2^{(W \times W)^*}$

Interpretation

Composite game relations

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Interpretation

- $R^i_{g_1 \cup g_2} = R^i_{g_1} \cup R^i_{g_2}$;
- $R^i_{g_1 ; g_2} = \{\mathfrak{f}(\mathfrak{T}(X); \mathcal{T}) \mid X \in R^i_{g_1}$ and
  $\mathcal{T} = \{\mathfrak{T}(X_1), \ldots, \mathfrak{T}(X_k)\}$ where $\{X_1, \ldots, X_k\} \subseteq R^i_{g_2}\}$;

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- $R^i_{g_1 \| g_2} = \{\mathfrak{f}(\mathfrak{T}(X_1) \| \mathfrak{T}(X_2)) \mid X_1 \in R^i_{g_1}$ and $X_2 \in R^i_{g_2}\}$.

A satisfiable formula

Consider the game trees T_4 and T_5.

A satisfiable formula

Consider the game trees T_4 and T_5.

Suppose $M, w \models \bigwedge_{a_j \in \{a,b\}} (\langle a_j \rangle True \wedge$
$[a_j](\wedge_{a_j \in \{a,b\}} \langle a_j \rangle (\bigwedge_{a_j \in \{c,d\}} (\langle a_j \rangle True \wedge [a_j](\wedge_{a_j \in \{c,d\}} \langle a_j \rangle True));$

and, $M, w \models T_4^{\checkmark}$.

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and, $M, w \models T_4^{\surd}$.

Then it follows that,

$M, w \models \langle T_4, 1 \rangle \alpha \to \langle T_4 \| T_5, 2 \rangle \alpha$.

Axiom system

- $\langle h, i \rangle \alpha \equiv$?

(Informally): Game h is enabled and player i has a strategy given by a sub-tree X of the enabled game such that *frontier*(X) satisfies α.

Axiom system

- $\langle h, i \rangle \alpha \equiv h^{\surd} \wedge \downarrow_{(h,i,\alpha)}$ (push)

(Informally): Game h is enabled and player i has a strategy given by a sub-tree X of the enabled game such that *frontier*(X) satisfies α.

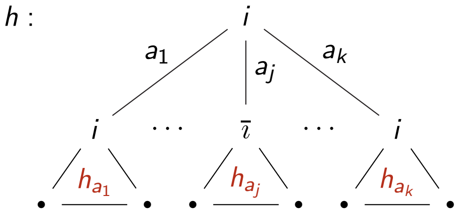
- $\langle a \rangle \alpha$ - can be encoded in the logic.
- $h^{\surd}$ can be defined.

Definition of *push*

h is a single node:

- $\downarrow_(h, i, \alpha) = \alpha$.

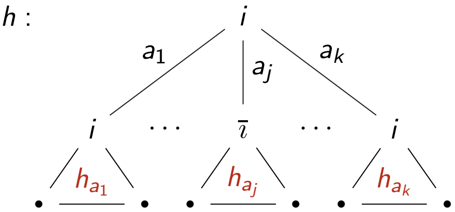
Axiom system



$\downarrow_{(}\ h, i, \alpha)$ holds at state u:

- if u is a player i node
  - then $\exists w$ such that $u \xrightarrow{a} w$ and $\langle h_a, i\rangle\alpha$ holds at w.

Axiom system



$\downarrow_{(}\ h, i, \alpha)$ holds at state u:

- if u is a player i node
  - then $\exists w$ such that $u \xrightarrow{a} w$ and $\langle h_a, i \rangle \alpha$ holds at w.

- if u is node taken by a player in $\bar{\imath}$
  - $\forall a_j$ such that $u \xrightarrow{a_j} w$, $\langle h_{a_j}, i \rangle \alpha$ holds at w.

Axiom system

General idea behind *push*

- If the root is a player i-node then
  - an edge is chosen and the requirement is "pushed" to the relevant subtree.
- If the root is a node taken by a player in $\bar{\imath}$ then
  - all outgoing edges need to be taken into account and the requirement is "pushed" to all the resulting subtrees.

Axiom system

- Propositional axioms:
  - All the substitutional instances of tautologies of PC.
  - $\mathbf{turn}_i \equiv \bigwedge_{j \in \bar{\imath}} \neg \mathbf{turn}_j$.
- Axiom for single edge games:
  - $\langle a \rangle (\alpha_1 \vee \alpha_2) \equiv \langle a \rangle \alpha_1 \vee \langle a \rangle \alpha_2$.
  - $\langle a \rangle turn_i \supset [a] turn_i$.
- Reduction axioms:
  - $\langle g_1 \cup g_2, i \rangle \alpha \equiv \langle g_1, i \rangle \alpha \vee \langle g_2, i \rangle \alpha$.
  - $\langle g_1 ; g_2, i \rangle \alpha \equiv \langle g_1, i \rangle \langle g_2, i \rangle \alpha$.
  - $\langle g_1 \| g_2, i \rangle \alpha \equiv \bigvee_{h \in init(g_1 \| g_2)} head_h^{\checkmark} \wedge comp^i(h, g_1, g_2, \alpha)$

Inference rules

$$(MP)\ \frac{\alpha, \quad \alpha \supset \beta}{\beta} \qquad (NG)\ \frac{\alpha}{[a]\alpha}$$

Parallel reduction axiom

Modelling interleaving of players.

Init and Residue

- Init: $init(g)$ = the initial (atomic) game of g.
- Residue: $g \backslash h$ = the game expression generated after playing the initial atomic game h in $init(g)$.

Parallel reduction axiom

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Init and Residue

- Init: $\mathit{init}(g) =$ the initial (atomic) game of g.
- Residue: $g \backslash h =$ the game expression generated after playing the initial atomic game h in $\mathit{init}(g)$.

Parallel reduction axiom: intuitive idea

- There exists an atomic tree $h \in \mathit{init}(g_1 \| g_2)$ such that $\mathit{head}(h)$ is enabled.
- Player i has a strategy in $\mathit{head}(h)$ which when composed with a strategy in the residue ensures α. We use $\mathit{comp}^i(h, g_1, g_2, \alpha)$ to denote this property.

Expressing $comp^i(h, g_1, g_2, \alpha)$

Suppose $h = (S, \Rightarrow, s_0, \widehat{\lambda})$, $A = moves(s_0) = \{a_1, \ldots, a_k\}$.

If $h \in init(g_1)$, $h \in init(g_2)$ and,

- $\widehat{\lambda}(s_0) = i$ then $comp^i(h, g_1, g_2, \alpha) =$
  $\bigvee_{a_j \in A}(\langle a_j\rangle\langle(h_{a_j}; (g_1 \backslash h))\| g_2\rangle\alpha \vee \langle a_j\rangle\langle g_1\|(h_{a_j}; (g_2 \backslash h))\rangle\alpha)$.
- $\widehat{\lambda}(s_0) \in \bar{\imath}$ then $comp^i(h, g_1, g_2, \alpha) =$
  $\bigwedge_{a_j \in A}([a_j]\langle(h_{a_j}; (g_1 \backslash h))\| g_2\rangle\alpha \vee [a_j]\langle g_1\|(h_{a_j}; (g_2 \backslash h))\rangle\alpha)$.

Decidability

A formula is satisfiable iff it is satisfiable in an exponential sized model.

Given α to decide if α is satisfiable:

- Guess an exponential sized model M.
- Explicitly build the relation $R_g^i \subseteq 2^{(W \times W)^*}$.
  - Time: exponential in the size of the model.
- Check whether M satisfies α.

Coalition logic

Language:

$\phi := \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid [C]\phi$

Coalition logic

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Coalitional model: $\mathcal{M} = (S, \{E_C \mid C \subseteq N\}, V)$, where
$E_C : S \longrightarrow \mathcal{P}(\mathcal{P}(S))$ is monotonic.

Coalition logic

Language:
$\phi := \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid [C]\phi$

Coalitional model: $\mathcal{M} = (S, \{E_C \mid C \subseteq N\}, V)$, where
$E_C : S \longrightarrow \mathcal{P}(\mathcal{P}(S))$ is monotonic.

$\mathcal{M}, s \models [C]\phi$ iff sE_CX, where $X = \{x \in S : \mathcal{M}, x \models \varphi\}$

$[C]\phi$ holds at a state s iff coalition C is effective for the truth-set of ϕ.

Which book to read?

- A father of 3 daughters would like to read a book on the relationship between sexes.
- He wants his daughters' opinion regarding which book to read.
- Simon de Beauvoir's "Le deuxième sexe" or Susan Faludi's "Backlash".
- The majority opinion will decide.

Which book to read?

$$\varphi = [\{1,2\}]b \wedge [\{2,3\}]b$$

Which book to read?

$\varphi = [\{1, 2\}]b \wedge [\{2, 3\}]b$

$S = \{s, b, f\}$

$E(s)$

	B	F
B	b	b
F	b	f

	B	F
B	b	f
F	f	f

Daughter 1 is the row player, 2 is the column player and 3 decides between the tables.

Model checking

Given $\mathcal{M}$ and ϕ, find $\phi^{\mathcal{M}}$

$\phi^{\mathcal{M}}$ can be calculated in time $\mathcal{O}(|\mathcal{M}| \times |\phi|)$

Axiomatics

Mon

- propositional tautologies,
- $$\frac{\varphi,\ \varphi \to \psi}{\psi}$$
- $$\frac{\varphi \to \psi}{[C]\varphi \to [C]\psi}$$

Satisfiability

Given a formula ϕ, does there exist a model $\mathcal{M}$ and a state s, such that $\mathcal{M}, s \models \phi$

Result : *Mon* is NP-complete.