An improved probability clock: addendum to the original OOM article from the year 2000

Herbert Jaeger, Feb 28, 2012

The article

H. Jaeger (2000): **Observable operator models for discrete stochastic time series.** Neural Computation 12(6), 2000, 1371-1398

describes an example of a stochastic process which can be modeled by an OOM but not by any HMM. This process, dubbed the *probability clock*, has been often cited in subsequent publications by the same or other authors. The original paper gives a generic specification of this process on pages 12ff. The idea is to design a 2-operator OOM where the first operator, τ_a, is a rotation of $\mathbb{R}^3$ around the x-axis by an angle φ, and the other operator τ_b is a projection on the starting vector $w_0 = (r, s, t)'$. In the paper above, values $\varphi = 1$ and $(r, s, t) = (0.75, 0, 0.25)$ were used. This setting indeed yields a valid non-HMM OOM. However, it has been pointed out to me by Mingjie Zhao that for some settings of φ, with $(r, s, t) = (0.75, 0, 0.25)$ no valid OOM is obtained, i.e. some "probabilities" computed by such models are negative.

It is remarkable that this imperfection has remained unnoticed for so many years...

A remedy is to use a different starting vector, for instance if one puts $(r, s, t) = (0.9, 0, 0.1)$, then one obtains a valid OOM for all settings of φ. Like in the original example, such OOMs are non-HMMs for all φ which are non-rational multiples of π.