Exercises for Comp & Comp, Spring 2019, Sheet 10 – Solutions

Please return Tuesdy May 7 in class.

Problem 1 (rather easy). Prove or disprove the following claim:

Let $R \subseteq \Sigma^* \times \Sigma^*$ be a polynomially decidable relation. Furthermore, assume that R is *constant balanced*, that is, there exists a constant C such that $(x, y) \in R$ implies $|y| \leq C$. Let $L = \{w \mid (w, y) \in R \text{ for some } y\}$. Then $L \in \mathbf{P}$.

Solution. Claim is true. Let $L = \{w \mid (w, y) \in R \text{ for some } y\}$. There are finitely many y with $|y| \leq C$, say, K many. They can be hardcoded in the transition function and state set of a deterministic TM which on input w systematically tries all y (makes K such tries) and for each y checks in polynomial time $O(n^k)$ whether $(w, y) \in R$. Total time is $O(Kn^k) = O(n^k)$.

Problem 2 (easy). Claim: SPACE(1) = TIME(1). Prove or disprove.

Solution. Claim is wrong. SPACE(1) (even SPACE(0)!) contains all regular languages, but TIME(1) doesn't because there are regular languages that need time n (e.g. the language of all words over {0,1} which end in 1: the TM has to wander its read head to the end of the input word to verify this, taking n steps).

Problem 3 (needs an idea, but is rather straightforward). The problem SET PACKING has instances consisting of a finite collection C of finite sets and of a positive integer $K \leq |C|$. The question to be decided is whether C contains at least K disjoint sets. – The problem CLIQUE has instances consisting of an undirected graph $G = (V, E)$ and a positive integer $K \leq |V|$. The question to be decided is whether G contains a *clique* of size at least K, that is, a subset $V' \subseteq V$ such that every two vertices in V' are joined by an edge in E. Reduce SET PACKING to CLIQUE. Don't forget to demonstrate that your reduction can be done in polynomial time!

Solution. Let C, K be an instance of SET PACKING. We construct a graph $G = (V, E)$ by putting $V = C$, and $\{v_1, v_2\} \in E$ iff $v_1 \cap v_2 = \varnothing$ (for $v_1, v_2 \in C$). Then obviously G contains a clique of size at least K iff C contains at least K disjoint sets, that is, we have reduced SET PACKING to CLIQUE. This reduction can be carried out in polynomial time: we have to carry out $|C|^2$ many pairwise comparisons to check for all $v_1, v_2 \in C$ whether $v_1 \cap v_2 = \varnothing$. [Actually a bit fewer because we don't have to consider the pairings v_i, v_i]. A brute-force comparison of two finite sets v_1, v_2 to check whether they are disjoint needs $|v_1|\,|v_2|$ many individual checks whether some element $e_1 \in v_1$ is identical to another element $e_2 \in v_2$. All in all, this boils down to $O(|C|^2 m^2)$ pairwise element comparisons, where m is the maximum size of sets $v \in C$. Each pairwise element comparison can be done in time $O(\log (C\, m))$. All in all, the reduction can be computed in time $O(|C|^3 m^3)$.