PSM Spring 2019, homework 3

1. Let $\Omega = \{a, b, c\}$. Define two σ-fields on Ω such that their union is not a σ-field.

2. Let $\Omega = \{a, b, c, d\}$. Let $S = \{1, 2, 3\}$. Construct a nontrivial σ-field $\mathcal{F}$ on S and a σ-field $\mathfrak{A}$ on Ω and a function $X : \Omega \to S$ such that X is $\mathfrak{A}$-$\mathcal{F}$-measurable (a σ-field is trivial if it contains just the empty set and the whole set).

3. Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ and $(\Omega_3, \mathcal{F}_3)$ be measurable spaces. If $f_1 : \Omega_1 \to \Omega_2$ and $f_2 : \Omega_2 \to \Omega_3$ are respectively $\mathcal{F}_1$-$\mathcal{F}_2$ and $\mathcal{F}_2$-$\mathcal{F}_3$-measurable functions, prove that $f_2 \circ f_1 : \Omega_1 \to \Omega_3$, where $f_2 \circ f_1(x) := f_2(f_1(x))$ is $\mathcal{F}_1$-$\mathcal{F}_3$-measurable.

4. Let $(S, \mathcal{F})$ be a measurable space and $\varphi : S \to S'$ a map. Show that

   $$\mathcal{F}' := \{B \subseteq S' \mid \varphi^{-1}(B) \in \mathcal{F}\}$$

   is a σ-field on S'.

5. Let $\varphi : S \to S'$ be a map and let $\sigma(\mathcal{G}')$ be a σ-field on S' generated by $\mathcal{G}'$. Show that $\varphi^{-1}(\sigma(\mathcal{G}')) = \sigma(\varphi^{-1}(\mathcal{G}'))$.

   Hint: make use of the previous fact 4. You can also use the fact (another easy exercise) that the preimage of a σ-field is a σ-sigma field, that is, if $\varphi : S \to S'$ is a map and $\mathcal{F}'$ a σ-field on S', then $\varphi^{-1}(\mathcal{F}')$ is a σ-field on S.

6. Show that the square function $\text{sqr} : \mathbb{R} \to \mathbb{R},\ \ x \mapsto x^2$ is $\mathfrak{B}(\mathbb{R})$-$\mathfrak{B}(\mathbb{R})$-measurable. Hint: use the previous problem.

Note: problems 1 – 4 and 6 are easy, problem 5 is a little tricky.