PSM Spring 2019, homework 3 — Solutions

1. Let $\Omega = \{a, b, c\}$. Define two σ-fields on Ω such that their union is not a σ-field.
2. Let $\Omega = \{a, b, c, d\}$. Let $S = \{1, 2, 3\}$. Construct a nontrivial σ-field $\mathcal{F}$ on S and a σ-field $\mathfrak{A}$ on Ω and a function $X : \Omega \to S$ such that X is $\mathfrak{A}$-$\mathcal{F}$-measurable (a σ-field is trivial if it contains just the empty set and the whole set).
3. Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ and $(\Omega_3, \mathcal{F}_3)$ be measurable spaces. If $f_1 : \Omega_1 \to \Omega_2$ and $f_2 : \Omega_2 \to \Omega_3$ are respectively $\mathcal{F}_1$-$\mathcal{F}_2$ and $\mathcal{F}_2$-$\mathcal{F}_3$-measurable functions, prove that $f_2 \circ f_1 : \Omega_1 \to \Omega_3$, where $f_2 \circ f_1(x) := f_2(f_1(x))$ is $\mathcal{F}_1$-$\mathcal{F}_3$-measurable.
4. Let $(S, \mathcal{F})$ be a measurable space and $\varphi : S \to S'$ a map. Show that

$$\mathcal{F}' := \{B \subseteq S' \,|\, \varphi^{-1}(B) \in \mathcal{F}\}$$

   is a σ-field on S'.
5. Let $\varphi : S \to S'$ be a map and let $\sigma(\mathcal{G}')$ be a σ-field on S' generated by $\mathcal{G}'$. Show that $\varphi^{-1}(\sigma(\mathcal{G}')) = \sigma(\varphi^{-1}(\mathcal{G}'))$.

   Hint: make use of the previous fact 4. You can also use the fact (another easy exercise) that the preimage of a σ-field is a σ-sigma field, that is, if $\varphi : S \to S'$ is a map and $\mathcal{F}'$ a σ-field on S', then $\varphi^{-1}(\mathcal{F}')$ is a σ-field on S.
6. Show that the square function $\text{sqr} : \mathbb{R} \to \mathbb{R},\ \ x \mapsto x^2$ is $\mathfrak{B}(\mathbb{R})$-$\mathfrak{B}(\mathbb{R})$-measurable. Hint: use the previous problem.

Note: problems 1 – 4 and 6 are easy, problem 5 is a little tricky.

Solutions

1. Let $\mathcal{F}_1 = \{\emptyset, \Omega, \{1, 2\}, \{3\}\}$ and $\mathcal{F}_2 = \{\emptyset, \Omega, \{2, 3\}, \{1\}\}$. These are σ-fields but their union is not, because the union is not closed under intersection.
2. There are several solutions. The best way to get one of them, I think, is to start from creating some X and $\mathcal{F}$, then set $\mathfrak{A} = X^{-1}(\mathcal{F})$. Let's do it: With $X : a \mapsto 1, b \mapsto 1, c \mapsto 2, d \mapsto 2$ and $\mathcal{F} = \{\emptyset, S, \{1, 2\}, \{3\}\}$ one gets $\mathfrak{A} = \{X^{-1}(\emptyset), X^{-1}(S), X^{-1}(\{1, 2\}), X^{-1}(\{3\})\} = \{\emptyset, \Omega, \Omega, \emptyset\} = \{\emptyset, \Omega\}$.
3. We have to show that $(f_2 \circ f_1)^{-1}(C) \in \mathcal{F}_1$ for all $C \in \mathcal{F}_3$. Consider some $C \in \mathcal{F}_3$. Then $B := (f_2)^{-1}(C) \in \mathcal{F}_2$ because f_2 is $\mathcal{F}_2$-$\mathcal{F}_3$-measurable, and $A := (f_1)^{-1}(B) \in \mathcal{F}_1$ because f_1 is $\mathcal{F}_1$-$\mathcal{F}_2$-measurable. Since $A = (f_2 \circ f_1)^{-1}(C)$ we have shown that that $f_2 \circ f_1$ is $\mathcal{F}_1$-$\mathcal{F}_3$-measurable.
4. We show that $\mathcal{F}'$ satisfies the three conditions of a σ-field.
   - $\varphi^{-1}(S') = S$ and $S \in \mathcal{F}$, hence $S' \in \mathcal{F}'$.
   - Let $B \in \mathcal{F}'$, that is, $\varphi^{-1}(B) \in \mathcal{F}$. Observing $\varphi^{-1}(B^{\mathsf{C}}) = (\varphi^{-1}(B))^{\mathsf{C}}$, we see that $B^{\mathsf{C}} \in \mathcal{F}'$.

- Let $(B_n)_{n=1,2,\ldots}$ be a sequence of elements of $\mathcal{F}'$, that is, $\varphi^{-1}(B_n) \in \mathcal{F}$ for all n. Then $\varphi^{-1}(\bigcup_n B_n) = \bigcup_n \varphi^{-1}(B_n) \in \mathcal{F}$, hence $\bigcup_n B_n \in \mathcal{F}'$.

5. We have to show (i) $\varphi^{-1}(\sigma(\mathcal{G}')) \subseteq \sigma(\varphi^{-1}(\mathcal{G}'))$ and (ii) $\sigma(\varphi^{-1}(\mathcal{G}')) \subseteq \varphi^{-1}(\sigma(\mathcal{G}'))$.

   (ii) is clear because $\varphi^{-1}(\sigma(\mathcal{G}')) = \sigma(\varphi^{-1}(\sigma(\mathcal{G}')))$ (because pre-images of σ-fields are σ-fields) and $\varphi^{-1}(\mathcal{G}') \subseteq \varphi^{-1}(\sigma(\mathcal{G}'))$.

   (i) is a little tricky. By the definition of generating σ-fields,

   $$\sigma(\varphi^{-1}(\mathcal{G}')) = \bigcap\{\mathcal{H} \,|\, \mathcal{H} \text{ is a } \sigma\text{-field on } S \text{ and } \varphi^{-1}(\mathcal{G}') \subseteq \mathcal{H}\}.$$

   Let $\mathcal{H}$ be a σ-field on S containing $\varphi^{-1}(\mathcal{G}')$. We have to show that $\varphi^{-1}(\sigma(\mathcal{G}')) \subseteq \mathcal{H}$. Consider the set $\mathcal{B} = \{B \subseteq S' \,|\, \varphi^{-1}(B) \in \mathcal{H}\}$. Since $\mathcal{G}' \subseteq \mathcal{B}$ and (according to the fact listed in problem 4) $\mathcal{B}$ is a σ-field on S', it follows that $\sigma(\mathcal{G}') \subseteq \mathcal{B}$. Hence, $\varphi^{-1}(\sigma(\mathcal{G}')) \subseteq \mathcal{H}$.

6. We have to show that for every $B \in \mathfrak{B}(\mathbb{R})$, $\mathrm{sqr}^{-1}(B) \in \mathfrak{B}(\mathbb{R})$. Using the fact shown in problem 5, it is enough to show that for every $G \in \mathcal{G}$, where $\mathcal{G}$ is a generator of $\mathfrak{B}(\mathbb{R})$, $\mathrm{sqr}^{-1}(G) \in \mathfrak{B}(\mathbb{R})$. For $\mathcal{G}$ we use the set of all closed intervals $[a, b]$ where $a < b$. But $\mathrm{sqr}^{-1}[a, b] = \emptyset$ if $b < 0$ and $\mathrm{sqr}^{-1}[a, b] = [\sqrt{\max\{0, a\}}, \sqrt{b}] \cup [-\sqrt{b}, -\sqrt{\max\{0, a\}}]$ if $b \geq 0$, which in both cases clearly is a set in $\mathfrak{B}(\mathbb{R})$.