Biomedical Signal Processing

Data Engineering Program, spring 2018

Exercise Sheet 2: Solutions

April 22, 2018

Exercise 1) Discrete Random Processes

1-1. Let $x[n]$ and $y[n]$ be stationary, uncorrelated random signals. Show if $w[n] = x[n] + y[n]$, then $m_w = m_x + m_y$ and $\sigma_w^2 = \sigma_x^2 + \sigma_y^2$.

a. The mean:

$$w[n] = x[n] + y[n]$$

$$m_w = E\{w[n]\} = E\{x[n] + y[n]\} = E\{x[n]\} + E\{y[n]\} = m_x + m_y$$

b. The variance:

$$\sigma_w^2 = E\{(w[n] - m_w)^2\} = E\{(w[n])^2\} - 2E\{m_w w[n]\} + E\{w[n]\}$$

$$= E\{(w[n])^2\} - 2m_w^2 + m_w^2 = E\{(w[n])^2\} - m_w^2 = E\{(x[n] + y[n])^2\} - (m_x + m_y)^2$$

$$= E\{(x[n])^2\} + 2E\{x[n]y[n]\} + E\{(y[n])^2\} - m_x^2 - 2m_x m_y - m_y^2$$

$E\{x[n]y[n]\} = E\{x[n]\}E\{y[n]\} = m_x m_y$, since $x[n]$ and $y[n]$ are uncorrelated. Therefore:

$$\sigma_w^2 = E\{(x[n])^2\} - m_x^2 + E\{(y[n])^2\} - m_y^2 = \sigma_x^2 + \sigma_y^2$$

1-2. Let $e[n]$ denote a white noise sequence and $s[n]$ denote a sequence that is uncorrelated with $e[n]$. Show that the sequence, $y[n] = s[n]e[n]$ is white (that is $\mathcal{E}\{y[n]y[n+m]\} = A\delta[m]$ where A is a constant).

$$E\{y[n]y[n+m]\} = E\{s[n]e[n]s[n+m]e[n+m]\}$$

$$= E\{s[n]s[n+m]e[n]e[n+m]\}$$

$E\{s[n]s[n+m]e[n]e[n+m]\} = E\{s[n]s[n+m]\}E\{e[n]e[n+m]\}$since $s[n]$ and $e[n]$ are uncorrelated, and assuming $E\{s[n]e[m]\} = 0$ for all n and m. Therefore:

$$E\{y[n]y[n+m]\} = \sigma_s^2 \sigma_e^2 \delta[m]$$

1-3. consider a random signal $x[n] = s[n] + e[n]$ where both $s[n]$ and $e[n]$ are independent, stationary random signals with autocorrelation functions $\varphi_{ss}[m]$ and $\varphi_{ee}[m]$, respectively.

a) Determine expressions for $\varphi_{xx}[m]$ and $\Phi_{xx}(e^{j\omega})$.

$$\phi_{xx}[m] = E\{x[n]x[n+m]\} = E\{(s[n]+e[n])(s[n+m]e[n+m])\}$$

$$= E\{(s[n]s[n+m])\} + E\{(e[n]e[n+m])\} + E\{(s[n]e[n+m])\} + E\{(e[n]s[n+m])\}$$

$s[n]$ and $e[n]$ are independent and stationary, hence:

$$\phi_{xx}[m] = \phi_{ss}[m] + \phi_{ee}[m] + 2E\{s[n]\}E\{e[n]s[n]\}$$

If we assume $e[n]$ has zero mean, then $\phi_{xx}[m] = \phi_{ss}[m] + \phi_{ee}[m]$.

Taking the Fourier transform of $\phi_{xx}[m] = \phi_{ss}[m] + \phi_{ee}[m]$, we get:

$$\Phi_{xx}(e^{j\omega}) = \Phi_{ss}(e^{j\omega}) + \Phi_{ee}(e^{j\omega})$$

b) Determine expressions for $\varphi_{xs}[m]$ and $\Phi_{xs}(e^{j\omega})$.

$$\phi_{xs}[m] = E\{x[n]s[n+m]\} = E\{(s[n]+e[n])s[n+m]\} = \phi_{ss} + E\{(e[n])\}E\{(s[n])\}$$

If we assume $e[n]$ has zero mean, then $\phi_{xs}[m] = \phi_{ss}[m]$.

Taking the Fourier transform of $\phi_{xs}[m] = \phi_{ss}[m]$, we get:

$$\Phi_{xx}(e^{j\omega}) = \Phi_{ss}(e^{j\omega})$$

c) Determine expressions for $\varphi_{xe}[m]$ and $\Phi_{xe}(e^{j\omega})$.

Similar to b)

$$\phi_{xe}[m] = \phi_{ee}[m]$$

$$\Phi_{xx}(e^{j\omega}) = \Phi_{ee}(e^{j\omega})$$

1-4. Consider a random process $x[n]$ that is the response of the LTI system shown in Figure P1-4. In the figure, w[n] represents a real zero-mean stationary white noise process with $\mathcal{E}\{w^2[n]\} = \sigma_w^2$.

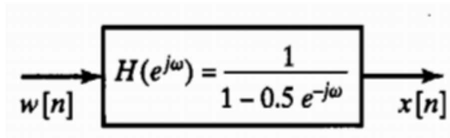


Figure P1-4.

a) Express $\mathcal{E}\left\{x^2[n]\right\}$ in terms of $\varphi_{xx}[n]$ and $\Phi_{xx}(e^{j\omega})$.

$$E\{x[n]x[n]\} = \phi_{xx}[0]$$

b) Determine $\Phi_{xx}(e^{j\omega})$, the power density spectrum of $x[n]$.

$$\begin{aligned}
\Phi_{xx}(e^{j\omega}) &= X(e^{j\omega})X^*(e^{j\omega}) \\
&= W(e^{j\omega})H(e^{j\omega})W^*(e^{j\omega})H^*(e^{j\omega}) \\
&= \Phi_{ww}(e^{j\omega})\,|\,H(e^{j\omega})\,|^2 \\
&= \sigma_w^2 \frac{1}{1 - cos(\omega) + \frac{1}{4}}
\end{aligned}$$

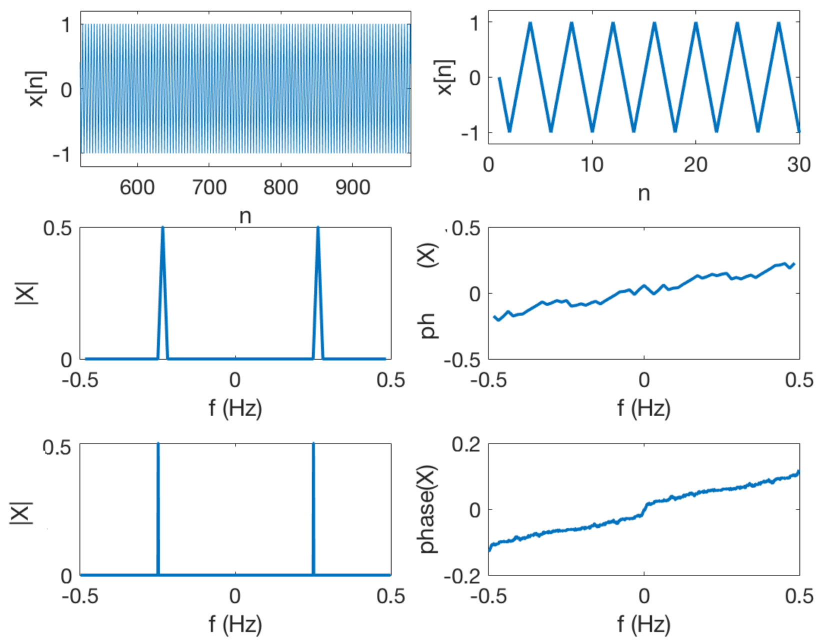
c) Determine $\varphi_{xx}[n]$, the correlation function of $x[n]$.

$$\begin{aligned}
\phi_{xx}[n] &= \phi_{ww}[n] * h[n] * h[-n] \\
&= \sigma_w^2\big((\frac{1}{2})^n u[n] * (\frac{1}{2})^{-n} u[-n]\big) \\
&= \sigma_w^2 \phi_{hh}[n]
\end{aligned}$$

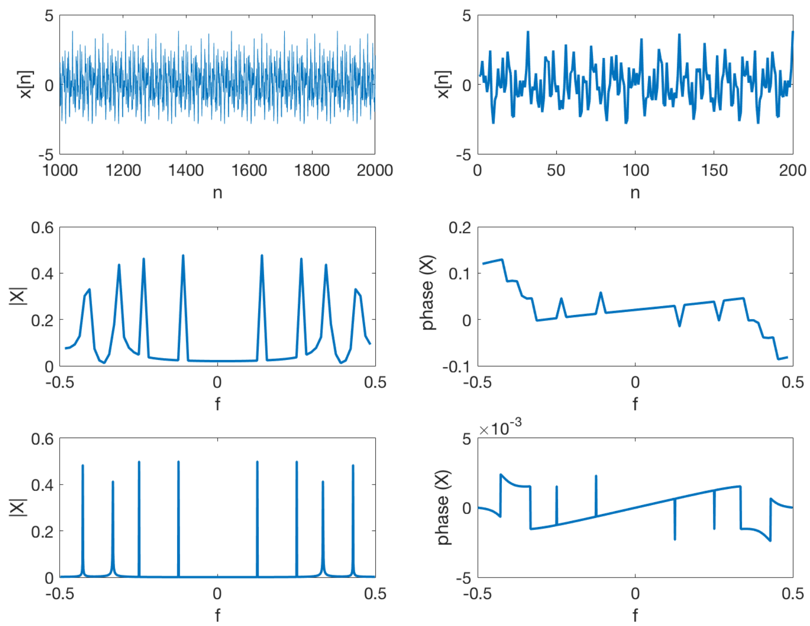
Exercise 2) Programming Exercise

For each signal, plots in the first row show the signal, in the middle row display 64-points FFT and the last row 1024-point FFT. We will discuss the results in the class.

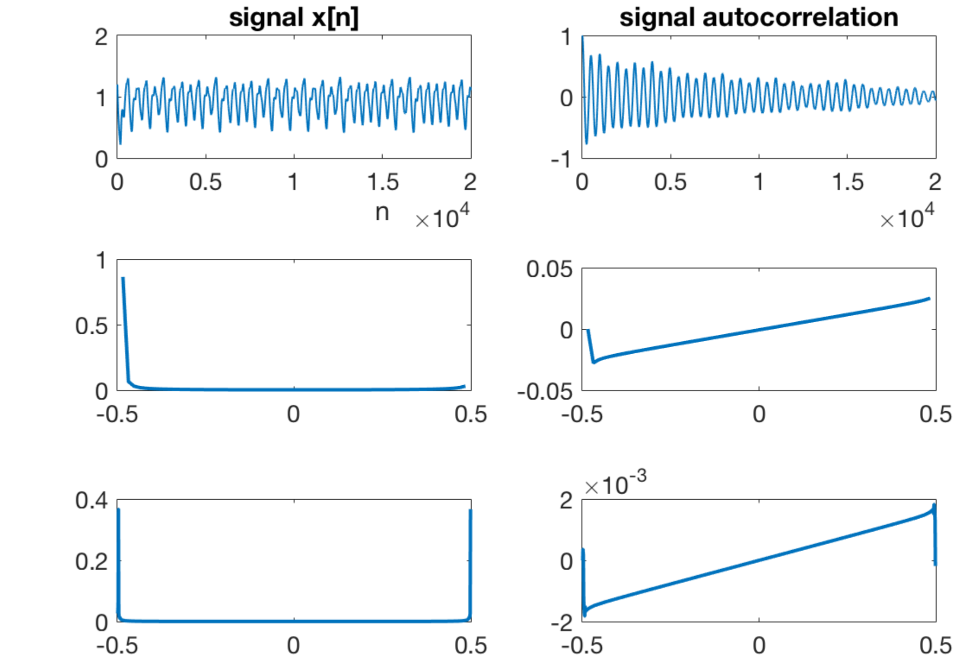
Signal 1)



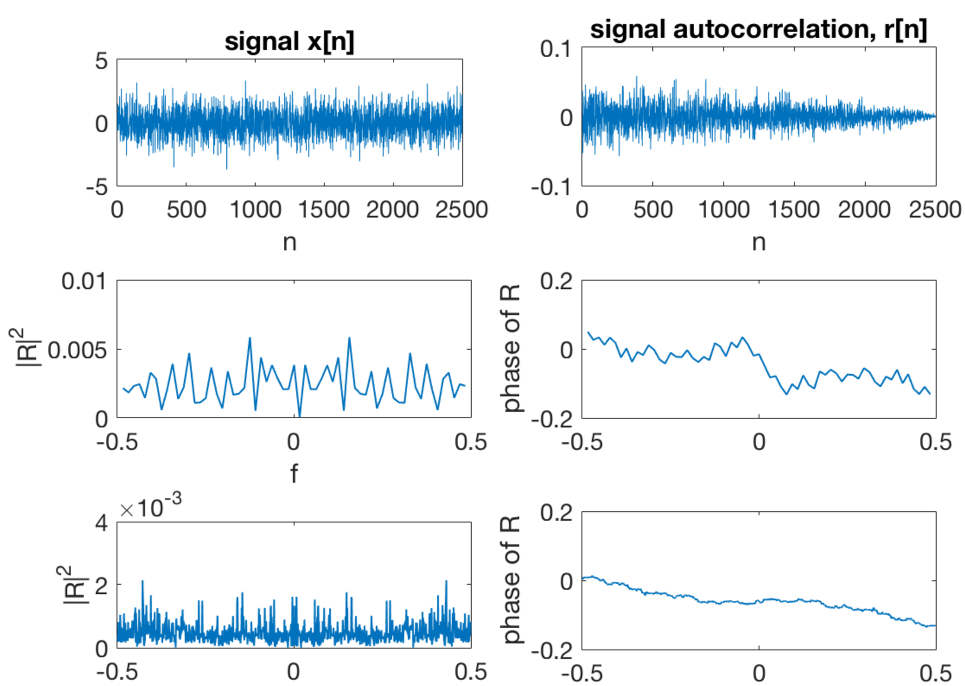
Signal 2)

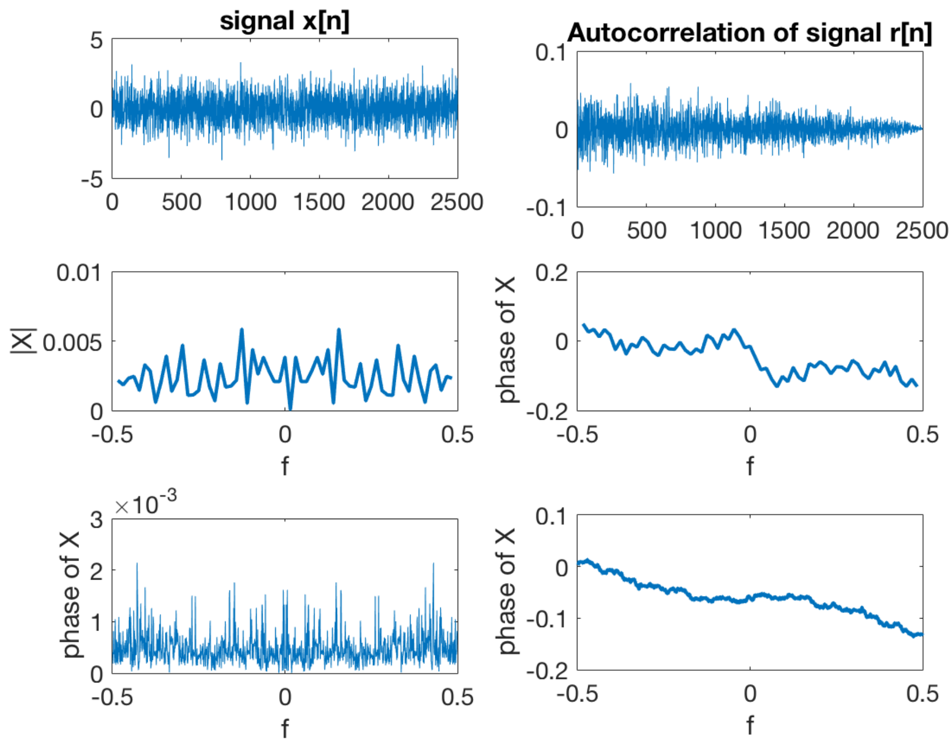


Signal 3)



Signal 4)





signal x[n]
Autocorrelation of signal r[n]
|X|
phase of X
f
phase of X
phase of X
f
f
f