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THE DYNAMICS OF RANDOM DIFFERENCE EQUATIONS IS REMODELED BY CLOSED RELATIONS*

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Abstract. We provide substantial simplifications in the understanding of qualitative dynamics of random difference equations (RDEs) by constructing closed relations using their typical instantiations. We consider the RDE given by $x_{n+1} = g(\xi_n(\omega), x_n)$, where $g : U \times X \to X$ is a uniformly continuous map, $\{\xi_n\}$ is a U-valued stationary input, and X a compact metric space. We represent each nonautonomous difference equation (NDE) $\{g_{n,\omega}(\cdot) := g(\xi_n(\omega), \cdot) : n \in \mathbb{Z}\}$ obtained by any realization of the input by some closed relation and prove a number of useful results. For all typical realizations of the input process, we show (1) any solution of the NDE can be obtained as an orbit (trajectory) of the corresponding *closed relation*; (2) every attractor A (an autonomous subset of X) of the closed relation contains a positively invariant uniform attractor and a pullback attractor of the NDE; (3) every (entire) solution of the NDE converges to an A in (2) or else stays in its dual repeller; (4) the closed relations are all identical whenever the input process is ergodic. Uniformity is a highly relevant condition for a nonautonomous attractor to be of practical relevance. Statement (3) yields a "Conley-like" decomposition theorem in terms of *autonomous subsets* alone and leads to a remarkable simplification in understanding the asymptotic dynamics of each such NDE.

Key words. random difference equation, stochastic difference equation, Conley decomposition theorem, nonautonomous dynamical systems, uniform attractors, iterated relations

AMS subject classifications. 37H99, 39A50, 37C70, 37B55, 37B25

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1. Introduction.

The existence of various types of attractors of dynamical systems which are subject to stochastic influence has been under investigation for the last two decades. Such studies falls into two broad categories, the first concerning the existence of a global attractor (e.g., [2, 6]) and the second concerning the local subattractor structure within a global attractor or that of a compact invariant set [17, 12, 7]. In this paper we engage in the latter task of a local analysis. [17, 12, 7] make an analysis for random flows. For random semiflows in continuous time, the problem was addressed in [13].

We carry out a local subattractor analysis of a subclass of random dynamical systems comprising discrete-time continuous systems, with the stochastic influence originating from a stationary source. Such systems arise naturally in many applications in the form of random difference equations (RDEs). However, the notion of a random dynamical system encompasses some more general situations, notably that of a nonstationary stochastic influence like Brownian motion (see [2]). For RDEs, our main concern is not to arrive at a Morse decomposition, i.e., the description of the structure of the attractor as a finite number of compact invariant sets and connecting orbits between them. Instead we replace typical instantiations of the RDE by a closed relation. This eventually provides us with an alternative and simpler picture of the asymptotic behavior of solutions using *autonomous* subsets of the state space (via a Conley kind of decomposition theorem), and also allows us to establish the

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existence of (nonautonomous) uniform local attractors. Uniformity of attractors is a strong condition ([4]; also see [3, 10]) which poses challenges to their analysis; a weaker notion of attractivity known as forward attractive per se does not provide convergence of solutions to an attractor component at any point of time, and hence in a practical situation where the attractor component is itself to be determined, this lack of convergence is a drawback. This problem is overcome with another weaker notion of pullback attractivity, but it per se does not capture the asymptotic behavior of solutions in the positive direction, and since pullback attractors need not be forward attractors and vice versa (see [10, Chap. 3]), the stronger notion of uniform attractors is a condition of significant practical relevance which makes its study worth some effort. Uniform attractors also point to a concept of the attractive strength of the attractor not deteriorating with time either in the forward or positive direction and have not been studied before for RDEs. Also, unlike some previous results such as in [17], which seem to be restricted to state spaces with normed topologies, we consider arbitrary compact metric spaces. This, for instance, permits us to consider state spaces with nonempty interior and homeomorphic to compact subsets of $\mathbb{R}^d$ $(d \leq \infty)$ under the product topology.

More specifically, we consider the RDE of the form

$$x_{n+1} = g(\xi_n(\omega), x_n), \tag{1}$$

where $g : U \times X \to X$ is a uniformly continuous map, U is a complete metric space, $\{\xi_n\}_{n\in\mathbb{Z}}$ is a stationary process defined on $(\Omega, \mathcal{F}, P)$ and taking values in U, and X is a compact metric space. The space X is chosen to be compact for two reasons here: first, our study is about local attractors, hence X could be treated as a global attractor or as a compact invariant set of an unbounded system; and second, there are many real world systems which warrant X to be compact. It is a practice to also call ξ_n the perturbation or noise. We prefer to call ξ_n in this paper the *input* to the RDE because this is more natural for the applications that the authors have encountered (see [14]). RDEs of the above nature arise in various situations, for instance, in financial market models, epidemic models, population biology, neural network models, discretization of stochastic dynamical systems, control theory, and many more. Also infinite-dimensional RDEs which arise when a difference equation with an unbounded random delay, that were previously analyzed only for their global behavior [8] due to the normed topology, can be considered here for local analysis since X is an arbitrary compact metric space.

Clearly, in (1), for each $\omega \in \Omega$, the sequence $\{\xi_n(\omega)\}$ generates a set of maps $\{g_{n,\omega}(\cdot) := g(\xi_n(\omega), \cdot)\}$ such that each $g_{n,\omega} : X \to X$ is continuous. For each ω in (1), we construct a closed relation [1, 15] on X which is a closed subset of $X \times X$. Like a map, a closed relation can be iterated to give rise to a dynamical system. We use the theory of iterating such relations to establish a number of results.

A fundamental theorem in dynamical systems (for flows and maps) is the Conley decomposition theorem [5] which provides for the decomposition of a flow or a map on a compact metric space into a part that exhibits a particular type of recurrence (called the chain-recurrent set) and a part in which the dynamics are essentially one-way (called the gradient part). Roughly speaking, a point is chain recurrent if it can return to itself by following the flow or iterates for an arbitrarily long time, given the liberty of making arbitrarily small jumps, or errors, along the way. We refer to the original definition of chain recurrence in [1, 16], but keeping in mind our objectives, it suffices for us to recall the resemblance between the chain-recurrent set and what are called the connecting orbits via the Conley decomposition theorem. The nature of this deep result remains alike for flows, maps, and also for closed relations [16].

We recall all relevant results concerning the dynamics of closed relations in section 2.2. Given an attractor A of a closed relation F on a compact space X, then the associated set of all "connecting orbits" with A is defined by $Basin(A) - A$. The terminology of a connecting orbit arises from the fact that for any $x \in Basin(A) - A$, and an orbit of F passing through x, the ω-limit set of the orbit is contained in A while its α-limit set is contained in the dual repeller of A. In the special case, where A and its dual repeller A^* are both singletons, the curious reader may note that a connecting orbit between them is popularly known as a heteroclinic orbit. Denoting the chain-recurrent set of the closed relation F by $\mathfrak{CR}(F)$, we recall the Conley decomposition theorem for closed relations.

THEOREM 1.1 (Conley decomposition theorem (see [16])). *Let F be a closed relation on a compact space X. Then the chain-recurrent set is equal to the union of the set of all connecting orbits associated with all of its attractors, i.e.,*

$$X - \mathfrak{CR}(F) = \bigcup_{C \in \mathcal{A}(F)} Basin(C) - C, \tag{2}$$

where $\mathcal{A}(F)$ is the set of all attractors of F.

Since the behavior of a connecting orbit is one-way, by the decomposition theorem, it follows that the dynamics on the chain-recurrent say is also one-way. In contrast, the chain-recurrent set not only captures all the asymptotic behaviors of all orbits, but the dynamics within in it includes all types of nontrivial, nonmonotonic behaviors. Thus, the chain-recurrent set and its complement give a dynamical decomposition. Also, further, since an attractor A is invariant under F, as a simple consequence (see Proposition 2.3) of this decomposition, it follows that if an attractor A is not entirely contained in $\mathfrak{CR}(F)$, then there exists a subattractor $A_0 \subsetneq A$. Thus the identification of the chain-recurrent set specifies whether an attractor contains smaller subattractor(s).

In this paper, we interpret the Conley decomposition of the closed relations constructed for each $\{g_{n,\omega}\}$ to delineate the asymptotics of the solutions of the NDEs. What differentiates our work from the previous results (see [12, 13]) is that the attractors of the closed relations and their basins being subsets of the state space alone suffice for this delineation, and not the traditionally employed nonautonomous subsets, i.e., subsets of $\mathbb{Z} \times X$. Our main results are stated via Theorem 4.1 and Theorem 5.1 which are proved post construction of the closed relation for the nonautonomous difference equations (NDEs) $\{g_{n,\omega}\}$. Here, to rigorously and quickly summarize these two theorems without going into the details of such a construction, we state a theorem without the explicit construction of the closed relations, but rather with an existential statement.

THEOREM 1.2. *Let $g : U \times X \to X$ define an RDE in* (1)*, with g also being a uniformly continuous map. Then for each ω belonging to a set of probability* 1*, there exists a closed relation $\widehat{F}(\omega)$ defined on a subspace $\widehat{X}(\omega)$ of X which satisfies the following:*

(i) *every entire-solution of $\{g_{n,\omega}\}$ is an orbit of $\widehat{F}(\omega)$ (see Definitions* 2.2 and 2.9*);*

(ii) *every nonempty attractor of $\widehat{F}(\omega)$ contains a (componentwise nonempty) local +invariant uniform attractor and a (componentwise nonempty) local pullback attractor (see Definition* 2.4*);*

(iii) *if $C(\omega)$ is an attractor of $\widehat{F}(\omega)$, and $\{\vartheta_n\}$ is an entire solution of $\{g_{n,\omega}\}$, then $\{\vartheta_n\}$ lies entirely either in $C(\omega)$ or $C^*(\omega)$ or else, the ω-limit set and and α-limit set of $\{\vartheta_n\}$ lie in $C(\omega)$ and $C^*(\omega)$, respectively, where $C^*(\omega)$ is the dual repeller of $C(\omega)$;*

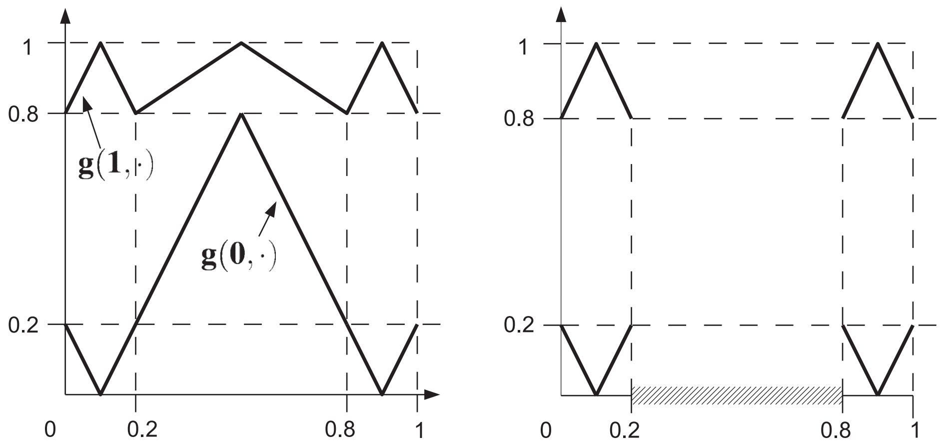
(iv) *if $C(\omega)$ is not contained entirely in $\mathfrak{CR}(\widehat{F}(\omega))$, then there exists at least one nonempty subattractor $C_0(\omega) \subsetneq C(\omega)$ of $\widehat{F}(\omega)$;*

(v) *if $\{\xi_n\}$ is ergodic, then the closed relations $\widehat{F}(\omega)$ on $\widehat{X}(\omega)$ that satisfy* (i)–(iv) *are all identical for all ω belonging to a set having probability* 1.

Statement (i) in Theorem 1.2 is significant since an entire solution of the NDEs obtained by a typical input realization can be related to the dynamical entities of the closed relation. Next, concerning statement (iii), if $\{\vartheta_n\}$ intersects $Basin(C(\omega)) - C(\omega)$, then it is a connecting orbit (of the concerned closed relation) between $C(\omega)$ and $C^*(\omega)$. Further, if $C(\omega)$ does not lie entirely inside $\mathfrak{CR}(\widehat{F}(\omega))$, then statement (ii) can be applied to the subattractor C_0 in (iv) to obtain finer (see proof of Theorem 4.1) local nonautonomous attractors contained in $C_0(\omega)$. Also, by applying (iii), we get to know where the limit sets of the entire solution lie with respect to $C_0(\omega)$. This gives a comprehensive picture of the inner structure of the dynamics inside $C(\omega)$ and further smaller dynamical entities can be found within any subattractor which does not lie within $\mathfrak{CR}(\widehat{F}(\omega))$. Our main contribution on nonautonomous attractors is on the existence of uniform attractors made in (ii). Statement (v) of Theorem 1.2, tells us that in the ergodic case, there is a *deterministic autonomous subset* which is the chain-recurrent set of the closed relation obtained by all typical NDEs that captures all the asymptotic behavior of the entire RDE with probability 1.

During the review of this paper, an anonymous referee posed the question as to whether the closed relation that can be obtained by the set-valued mapping $g(U, \cdot) : X \to 2^X$ is in any way related to a closed relation that may satisfy (i)–(iv) of Theorem 1.2. Since the idea behind the construction of the closed relation that satisfies Theorem 1.2 is made only in the later sections of this paper, and could be surrounded by other technicalities, we clarify here that the set-valued mapping $g(U, \cdot)$ (or the closed relation it could generate) would in general fail to shed light on the dynamics of the RDE or, more particularly, the dynamics of an NDE $\{g_{n,\omega}\}$ obtained from a typical realization of $\{\xi_n\}$. During the course of this clarification, we also provide some intuition on the construction of the closed relation whose existence was stated in Theorem 1.2.

The set-valued map $g(U, \cdot)$ generates a closed relation defined on the entire space X, and when $g(u, \cdot)$ for some $u \in U$ is found to be dissipative, i.e., $g(u, X)$ is a proper subset of X, the generated closed relation could fail to capture the dissipative dynamics involving the map $g(u, \cdot)$. The closed relation which ignores such dissipative dynamics of an NDE may not shed any light on the attractor dynamics of the NDE. We first illustrate this fact with a simple academic example. Consider the map $g(u, x)$ on the state space $X = [0, 1]$ and $U = \{0, 1\}$ defined by the two piecewise linear self-maps $g(0, \cdot)$ and $g(1, \cdot)$ on $[0, 1]$ as shown in Figure 1(a). To define an RDE, let $\{\xi_n\}$ be a U-valued independently and identically distributed process such that all typical realizations of $\{\xi_n\}$ contain both the elements 0 and 1 occurring infinitely often. The set-valued map $g(U, \cdot)$ generates a closed relation which is the union of the graphs of the functions $g(0, \cdot)$ and $g(1, \cdot)$ (see Figure 1(a)). Further, the only attractor of this closed relation (for definitions, see section 2.2) is the space $[0, 1]$ itself. That said about this set-valued map, let us analyze the dynamics of an NDE $\{g_{n,\omega}\}$ obtained from a typical realization $\{\xi_n(\omega)\}$. In all future dynamics following the time instants at which



(a) Closed relation defined on $[0, 1]$ via piecewise linear functions $g(0, \cdot)$ and $g(1, \cdot)$ (shown in thick lines).

(b) Closed relation defined on the base-domain $[0, 0.2] \cup [0.8, 1]$: Graphs of $g(0, \cdot)$ and $g(1, \cdot)$ restricted to the base-domain yield together the base-relation.

FIG. 1. *Example to illustrate the utility of the base domain.*

a 1 is followed by a 0 in a realization $\{\xi_n(\omega)\}$, the dynamics of the NDE $\{g_{n,\omega}\}$, is contained in the set $[0, 0.2] \cup [0.8, 1]$ since $g(1, [0, 1]) = g(1, [0, 0.2]) = g(1, [0.8, 1]) = [0.8, 1]$ and $g(0, [0.8, 1]) = [0, 0.2]$. In fact, once the dynamics is confined to the set $[0, 0.2] \cup [0.8, 1]$, an appearance of 0 as the input will constrain the entire solutions of any such NDE evaluated at the next time step to be contained in $[0, 0.2]$, while an appearance of 1, will set the entire solutions in the following time step to be contained in $[0.8, 1]$. The union of subintervals $[0, 0.2] \cup [0.8, 1]$ turns out to be what we call the base domain of the NDE (Definition 2.5). In general, when U is not necessarily finite, a rather special sequence of subsets $\{X_n \subset X\}$ of an NDE can be described such that the set of all entire solutions of the NDE evaluated at a time n_0 is identical to the set X_{n_0} (see Lemma 2.2); the set of accumulation points of such a sequence $\{X_n\}$ (in an appropriate topology) is called the base domain (Definition 2.5) of the NDE. It turns out that the base domain is a certain maximal invariant subset of X which is found to contain all the asymptotic dynamics of the corresponding NDE (see (ii) of Proposition 2.2).

Returning to the example in Figure 1(a), the union of the graphs of the functions $g(1, \cdot)$ and $g(0, \cdot)$ when restricted to the base domain $[0, 0.2] \cup [0.8, 1]$, is called the base relation of a typical NDE $\{g_{n,\omega}\}$ (see Figure 1(b)). In general, when U is not necessarily finite, the base relation on the base domain is obtained as the set of accumulation points of relations (in an appropriate topology) which are graphs of $\{g_{n,\omega}\}$ restricted to the base domain (Definition 2.6) of the NDE. The base relation is then found to capture the asymptotic dynamics of the corresponding NDE (see (iii) of Proposition 2.2). The point to be taken from this example is that the set-valued map $g(U, \cdot)$ provides no insight into the dynamics since its only attractor is the entire space, while all the base relation's attractors will be contained in the base domain.

We point out that the definition of base domain or base relation is not confined to academic examples. Suppose that in an RDE $g : U \times X \to X$, the family of maps $\{g(u, \cdot) : u \in U\}$ is uniformly dissipative in the sense that there is a proper subset Y of X independent of $u \in U$ so that $g(u, X) \subset Y$ for every $u \in U$. In such a case, the definition of a base domain is very relevant as the base domain would be no larger than Y. One such scenario familiar to the authors is a widely used machine learning application where a recurrent neural network comes with a hyperbolic tangent (tanh) nonlinearity (e.g., [14]) that defines an RDE: to illustrate our point, we consider only a one-dimensional network, i.e., $g(u,x) = \tanh(au + bx)$ defined on $U \times [-1,1] \to [-1,+1]$, where U is any compact subset of the real line and a, b are real numbers. Regardless of the U-valued stochastic process chosen, owing to the compactness of U, and to the fact that $\tanh([-1,1])$ is a proper subset of $[-1,1]$, the family $\{g(u,\cdot) : u \in U\}$ can be found to be uniformly dissipative. As an observation from this example, we remark that for dissipation to occur, $\{g(u,\cdot) : u \in U\}$ need not have to be a family of contraction maps, since the constant b can be made larger than 1 to yield a family of locally expanding maps.

The remainder of the paper is organized as follows. Sections 2 and 3 prepare the necessary background and results to state our main results. The main theoretical results, Theorems 4.1 and 5.1 are proved in sections 4 and 5, respectively. Finally, in section 6 we indicate where some of our results stand amongst others in the literature and also as to how they can be made use of.

2. Background and preliminaries: Formalizing the nonrandom case. Given any metric space (X,d), and any nonempty subset A, we denote $B_\eta(A) := \{y \in X : d(x,A) < \eta\}$ as an η-neighborhood of A. For nonempty sets $A, B \subset X$ we let $dist(A,B) := \sup\{d(x,B) : x \in A\}$ be the Hausdorff semidistance of A and B. In addition if any one of A and B is empty, we adopt the convention that $dist(A,B) = 0$. Subsets of $\mathbb{R}^d$ are endowed with the subspace topology induced from the standard topology of $\mathbb{R}^d$. Throughout the paper we use the notion of ω-limit of sets, i.e., the set of accumulation points of a sequence as time tends to infinity. Let $\{C_n\}$ be an (infinite or bi-infinite) sequence of subsets of a compact metric space X. We refer to the subset equivalently defined by $\limsup_{n\to\infty} C_n := \bigcap_n \overline{\bigcup_{k\geq n} C_k}$ or $\limsup_{n\to\infty} C_n := \{x \in X : \exists\ x_{n_k} \in C_{n_k} \text{ such that} \lim_{k\to\infty} x_{n_k} = x\}$ or $\liminf_{n\to\infty} d(x, C_n) = 0$ as the lim sup of the sequence $\{C_n\}$. To make obvious that $\limsup_{n\to\infty} C_n$ is related to the notion of the ω-limit of the sequence C_n, one can show for any $k \in \mathbb{Z}$, $\overline{\bigcup_{n\geq k} C_n} = \bigcup_{n\geq k} C_n \cup \limsup_{n\to\infty} C_n$.

The following elementary result is used repeatedly.

LEMMA 2.1. *Let (X,d) be a complete metric space. Suppose $\{C_i\}_{(i\in\mathbb{Z} \text{ or } \mathbb{N})}$ is a sequence such that $C_i \subset X$, and $C := \limsup_{n\to\infty} C_n$, then $\lim_{n\to\infty} dist(C_n, C) = 0$. Also, if $C_{i+1} \subset C_i$ for all i, then $C = \bigcap_n C_n$.*

The equation $x_{n+1} = g_n(x_n)$, where for each $n \in \mathbb{T}$ and $\mathbb{T} = \mathbb{Z}$, $g_n : X \to X$ is continuous with X being a (nonvoid) metric space is called an NDE. Alternate designations of an NDE in this paper are "$\{g_n : X \to X\}$," or just "$\{g_n\}$ on X" with other assumptions implicit. We recall a standard convenient formulation of nonautonomous systems.

DEFINITION 2.1. *Let $\mathbb{T}^2_{\geq} := \{(n,m) : n,m \in \mathbb{T}$ and $n \geq m\}$, where $\mathbb{T}$ is subset of $\mathbb{R}$. A* process *ϕ on a state space X is a continuous mapping $\phi : \mathbb{T}^2_{\geq} \times X \to X$ which satisfies the evolution properties:*

(i) *$\phi(m,m,x) = x$ for all $m \in \mathbb{T}$ and $x \in X$;*
(ii) *$\phi(n,m,x) = \phi(n,k,\phi(k,m,x))$ for all $m,k,n \in \mathbb{T}$ with $m \leq k \leq n$ and $x \in X$.*

Since we consider discrete-time dynamics, always $\mathbb{T} = \mathbb{Z}$, but we retain the notation $\mathbb{T}$ to symbolize time. It is readily observed that an NDE $\{g_n\}$ on X generates a process ϕ on X by setting: $\phi(m,m,x) := x$ and $\phi(n,m,x) := g_{n-1} \circ \cdots \circ g_m(x)$. To verify that ϕ is a process we need to verify continuity. The notion of continuity in the first two variables of ϕ becomes trivial. Also, the composition of finitely many continuous mappings makes the map $x_m \mapsto \phi(n, m, x_m)$ continuous, and hence ϕ is continuous.

Also, in the other direction, when $\mathbb{T} = \mathbb{Z}$, for every process ϕ on X, there exists an NDE $\{g_n\}$ on X defined by $g_n(x) := \phi(n+1, n, x)$. Henceforth, we use the terminology of a "process" while referring to an NDE and vice versa for convenience. Moreover all definitions made for a "process" have a natural extension to an NDE and vice versa.

A sequence of sets bearing a time index $n \in \mathbb{T}$ are called nonautonomous sets. The following nonautonomous sets will be frequently used.

DEFINITION 2.2. *Let ϕ be a process on a space X.*

(i) *An* entire solution *of a process ϕ is a sequence $\vartheta = \{\vartheta_n\}_{n\in\mathbb{Z}}$ such that $\vartheta_m \in X$ for all m and $\phi(n, m, \vartheta_m) = \vartheta_n$ for all $m \le n$.*

(ii) *A nonautonomous set $\mathcal{A} = \{A_n : A_n \subset X\}_{n\in\mathbb{Z}}$ is said to be* ϕ-invariant *if $\phi(n+1, n, A_n) = A_{n+1}$ for all n.*

(iii) *A nonautonomous set $\mathcal{A} = \{A_n : A_n \subset X\}_{n\in\mathbb{Z}}$ is said to be ϕ* positively invariant *or* ϕ +invariant *if $\phi(n+1, n, A_n) \subset A_{n+1}$ for all n.*

Clearly, an entire solution is a ϕ-invariant set, and a ϕ-invariant set is a ϕ +invariant set. Next, we develop some new terminology with the aim of specifying the attractive domains of the attractors when the maps in the NDE are not necessarily surjective.

For a given NDE, suppose if g_{n_0} is not surjective for some n_0, then $\{\vartheta_{n_0+1} : \{\vartheta_n\}$ is an entire solution $\}$ is a proper subset of X. Since the system states or the states of all possible entire solutions at a particular time instant may be a proper subset of X, it is useful to consider a subset X_n of X at each n which contains all possible states reachable at n. To do that we define the following subsets of X. Let $i < n$. Denote

$$X_{n,i} := g_{n-1} \circ \cdots g_{i+1} \circ g_i(X). \tag{3}$$

The set $X_{n,i}$ being the image of a finite composition of continuous maps is compact whenever X is compact. Also $X_{n,i}$ is nonempty. Further, $X_{n,i} \supset X_{n,i-1}$. Hence $\bigcap_{i<n} X_{n,i}$ is a nested intersection of closed nonempty subsets, and whenever X is compact, the intersection is nonempty.

PROPOSITION 2.1. *Given an NDE $\{g_n : X \to X\}$, where X is compact, define $X_n := \bigcap_{i<n} X_{n,i}$. Then $g_n(X_n) = X_{n+1}$ for all $n \in \mathbb{T}$.*

Proof. Let $A_1, A_2, \ldots,$ be a collection of nonempty subsets of X such that $A_{i+1} \subset A_i$ and g be a continuous function on X. Then from elementary arguments we can prove

$$g(A) = \bigcap_{i=1}^{\infty} g(A_i), \text{ where } A := \bigcap_{i=1}^{\infty} A_i. \tag{4}$$

We recall that by its definition, X_n is a nested intersection, and deduce

$$\begin{aligned} X_n &= \bigcap_{i<n} X_{n,i}, \\ g_n(X_n) &\overset{(4)}{=} g_n \circ g_{n-1}(X) \cap g_n \circ g_{n-1} \circ g_{n-2}(X) \cap \cdots \\ &= X_{n+1,n-1} \cap X_{n+1,n-2} \cap \cdots \\ &= X_{n+1,n} \cap X_{n+1,n-1} \cap X_{n+1,n-2} \cap \cdots \quad (\text{since } X_{n+1,n} \supset X_{n+1,n-1}) \\ &= X_{n+1}. \end{aligned}$$

∎

Based on Proposition 2.1 we define a nonautonomous set as a natural association for an NDE or for a process. We make these definition in a more general form.

DEFINITION 2.3. *An* association *of an NDE* $\{g_n : X \to X\}$ *is a sequence of subsets* $\{X_n\}$*, such that each* X_n *is nonempty, closed, and* $g_n(X_n) = X_{n+1}$ *for all* $n \in \mathbb{T}$*. In particular, an association* $\{X_n\}$ *is said to be* natural *if* $X_n := \bigcap_{i<n} X_{n,i}$*.*

An association *for a process* ϕ *on* X *is a sequence of subsets* $\{X_n\}$*, such that* X_n *is nonempty and closed, and* $\phi(n, m, X_m) = X_n$ *for all* $n \geq m$*. The association* $X_n = \bigcap_{m<n} \phi(n, m, X)$ *is called the* natural association *of* ϕ*.*

Clearly, if each g_n of an NDE is surjective on each X, then the natural association $\{X_n\}$ will be such that $X_n \equiv X$ for all n.

A helpful consequence of defining a natural association is that it forms a tight envelope around the entire solutions of the process (see (iii) of Lemma 2.2). The proof of (ii) in Lemma 2.2 is known (e.g., [10]).

LEMMA 2.2. *Given a process* ϕ *on a compact space* X*, let* $\{X_n\}$ *be its natural association. Then,* (i) *for all* $n \geq m$*,* $\phi(n, m, X_m) = X_n$*;* (ii) *a sequence* $\mathcal{A} = \{A_k\}$ *is* ϕ*-invariant if and only if for every pair* $k \in \mathbb{T}$*,* $x \in A_k$ *there exists an entire solution* $\{\vartheta_n\}$ *such that* $\vartheta_k = x$ *and* $\vartheta_k \in A_k$ *for all* $k \in \mathbb{T}$*;* (iii) *an association* $\{Y_n\}$ *is the natural association of* ϕ*, i.e.,* $Y_n = X_n$ *for all* n *if and only if every entire solution* $\{\vartheta_n\}$ *is such that* $\vartheta_k \in Y_k$ *for all* $k \in \mathbb{T}$*.*

Proof. (i) By equating $X_{n,i} = \phi(n, i, X)$ in Proposition 2.1, (i) follows.

(ii) (from [10]) ($\Longrightarrow$) Let $k_0 \in \mathbb{T}$ and choose $x \in A_{k_0}$. For $k \geq k_0$, define the sequence $\vartheta_k := \phi(k, k_0, x)$. Then by ϕ-invariance, $\vartheta_k \in A_k$ for any $k > k_0$. On the other hand, $A_{k_0} = \phi(k_0, k, A_k)$ for $k \leq k_0$ and so there exists a sequence $x_k \in A_k$ with $x = \phi(k_0, k, x_k)$ and $x_k = \phi(k, k-1, x_{k-1})$ for all $k \leq k_0$. Then define $\vartheta_k := x_k$ for $k \leq k_0$. This completes the definition of the entire solution ϑ_k.

($\Longleftarrow$) Suppose for any $k \in \mathbb{T}$ and $x \in A_k$, there is an entire solution $\{\vartheta_n\}$ satisfying $\vartheta_k \in A_k$ for all $k \in \mathbb{T}$. This implies $\phi(k+j, k, x) \subset A_{k+j}$ for all $j \geq 0$. Hence we have $\phi(k+j, k, A_k) \subset A_{k+j}$. The other inclusion follows from the fact that $\phi(k, k-j, \vartheta_{k-j}) = x$ for all $j \geq 0$.

(iii) ($\Longrightarrow$) $\{X_n\}$ is ϕ invariant from (i). From (ii), if there is an $x \in X_k$, then there is an entire solution $\{\vartheta_n\}$ such that $x = \vartheta_k$. ($\Longleftarrow$) Let $\{\vartheta_i\}$ be an entire solution. Now consider some ϑ_n. By definition there exists $x_k \in X$ such that $\phi(n, k, x_k) = \vartheta_n$ for all $k < n$. Clearly, $\phi(n, k, x_k) \in \phi(n, k, X)$ for all $k < n$. This implies $\vartheta_n \in \bigcap_{k<n} \phi(n, k, X) = X_n$. Since n was chosen arbitrarily, $\vartheta_i \in X_i$ for all i. ∎

Given a process ϕ and an association $\{X_n\}$, we adopt the following notation: for every $A \subset X_i$, we denote

$$B_\eta^{(i)}(A) := B_\eta(A) \cap X_i := \{x \in X_i : d(x, A) < \eta\}.$$

Visibly, the set $B_\eta^{(i)}(A)$ is dependent on the association for the process. We formally state our definition of various types of local attractors for a process. We incorporate the association into the attractor definition so that the domain of attraction is the solutions of the NDE.

DEFINITION 2.4. *Let* ϕ *be a process on a space* X*, with an association* $\{X_n\}$ *and* $\mathcal{A} = \{A_n\}$ *be a* ϕ *+invariant set such that each* A_n *is compact and* $\subset X_n$*. If for some* $\eta > 0$*, any of the following conditions*

$$\lim_{j\to\infty} dist(\phi(n, n-j, B_\eta^{(n-j)}(A_{n-j})), A_n) = 0 \text{ for all } n, \tag{5}$$

$$\lim_{j\to\infty} dist(\phi(n+j, n, B_\eta^{(n)}(A_n)), A_{n+j}) = 0 \text{ for all } n, \tag{6}$$

$$\lim_{j\to\infty}\sup_n dist(\phi(n,n-j,B_\eta^{(n-j)}(A_{n-j})),A_n)=0, \tag{7}$$

$$\lim_{j\to\infty}\sup_n dist(\phi(n+j,n,B_\eta^{(n)}(A_n)),A_{n+j})=0, \tag{8}$$

holds, then in that order, $\mathcal{A}$ is, respectively, called a local +invariant pullback attractor, local +invariant forward attractor, local +invariant uniform pullback attractor, and local +invariant uniform forward attractor with respect to the association $\{X_n\}$. A local +invariant uniform pullback attractor when it is simultaneously a local +invariant uniform forward attractor, is simply called a local +invariant uniform attractor. If in addition $\mathcal{A}$ is ϕ-invariant then they are also local invariant attractors of their types.

Trivially, any association $\{X_n\}$ itself is both a ϕ-invariant local pullback and a forward attractor w.r.t. $\{X_n\}$. In contrast to our definition of ϕ-invariant local attractors above, the local attractor in the literature such as in [10] is made independent of any of the associations for ϕ. In this literature, a ϕ-invariant local attractor of any of the type viz., pullback, forward, uniform pullback, and uniform forward is gotten by replacing $B_\eta^{(*)}(\diamond)$ with $B_\eta(\diamond)$, where $B_\eta(\diamond)$ is an η-neighborhood of $\diamond$ in X in the identities (5) to (8). As a technicality, the reader may note that the definition of local attractors permits them to be a sequence of empty sets, i.e., each A_n is empty in $\mathcal{A}$. There can be cases of +invariant attractors where each A_n can be empty for some $n < n_0$ and nonempty for all $n \geq n_0$, whereas in the case of ϕ-invariant attractors either all A_n are empty or nonempty.

The remainder of this section is presented in two subsections. In the first subsection, we associate a closed relation with each process. In the following subsection, we recall the results on iterations of closed relations.

2.1. Closed relation for a process. A closed relation on a space X is a closed subset of $X \times X$. In this subsection, we associate a closed relation with each given process or an NDE and its association.

Motivation. Consider the map $g(u,x) := 4ux(1-x)$ on $[0,1]$, $u \in \mathbb{R}$. The function $4x(1-x)$ (the full logistic map) has $\{0\}$ as a fixed point. Any input sequence u_n generates an NDE $g_n(\cdot) = g(u_n,\cdot)$. If it happens that $u_{n_0} = 0$ for some n_0, then for any association $\{X_n\}$ of the NDE, $X_n = \{0\}$ for all $n > n_0$. Thus the forward asymptotic dynamics of such an NDE becomes insensitive to any nonzero input u_n for $n > n_0$. To characterize this asymptotic behavior more generally, one can say that the asymptotic behavior of the NDE is contained in the ω-limit of $\{X_n\}$, i.e., the $\limsup$ of $\{X_n\}$ which happens to be $\{0\}$. In general, given an NDE $\{g_n\}$ and an association $\{X_n\}$, its forward asymptotic dynamics is tied to the ω-limit set or the $\limsup$ of $\{X_n\}$. We give a name to this set.

DEFINITION 2.5. *Given an NDE $\{g_n\}$ on X and an association $\{X_n\}$, the subset of X given by $\widehat{X} = \limsup_{n\to\infty} X_n = \bigcap_n \overline{\bigcup_{k>n} X_k}$ is called the* base domain *of the NDE w.r.t. $\{X_n\}$.*

Given an association $\{X_n\}$, it is not clear whether an X_{n_0} is contained in its base domain. We will find later in section 4 that the most natural condition we would be encountering is that each X_n is contained in $\widehat{X}$. As a simple example illustrating the definition of base domain, consider the map $g(u,x) = u\sqrt{|x|}$, where $u \in U = \{-1,+1\}$ and $x \in [-1,+1]$. Clearly, a sequence $\{u_n\} \subset U$ gives rise to an NDE. For instance, for the sequence $u_n := (-1)^n$, the natural association is $X_n = [0,1]$, when n is an odd integer and $X_n = [-1,0]$ when n is even. The base domain $\widehat{X}$ is the set $[-1,1]$, which is the union of X_n's. We observe later in the paper that the typical case is when each component of the association is contained in the base domain for any NDE derived from an RDE (Lemma 4.4).

To explain the role of the individual maps in the description of the (forward asymptotic) dynamics we define the following.

DEFINITION 2.6. *Given an NDE $\{g_n\}$ on X, and an association $\{X_n\}$, we define F_n as an* individual relation *on X_n by*

$$F_n := \Big\{(x, g_n(x)) : x \in X_n\Big\}.$$

Further we define the base relation $\widehat{F}$ *on its base domain* $\widehat{X}$ *by*

$$\widehat{F} := \limsup_{n\to\infty} F_n := \limsup_{n\to\infty} \Big\{(x, g_n(x)) : x \in X_n\Big\}. \tag{9}$$

Clearly, by definition, $\widehat{F}$ is a closed relation on $\widehat{X}$. The process-based version of the above definitions is given below.

DEFINITION 2.7. Given a process ϕ and an association $\{X_n\}$, (i) the subset of X given by $\widehat{X} := \limsup_{n\to\infty} X_n$ is called its *base domain*; (ii) the individual relations of ϕ on X_n are $F_n := \{(x, \phi(n+1, n, x)) : x \in X_n\}$; (iii) its base relation $\widehat{F}$ is the relation on its base domain $\widehat{X}$ defined by $\widehat{F} = \limsup_{n\to\infty} F_n := \limsup_{n\to\infty}\{(x, \phi(n+1, n, x)) : x \in X_n\}$.

PROPOSITION 2.2. *Given a process ϕ on a compact X, an association $\{X_n\}$ with base domain $\widehat{X}$, and base relation $\widehat{F}$, the following assertions hold:* (i) *$\widehat{X}$ and $\widehat{F}$ are nonempty subsets of X and $X \times X$, respectively;* (ii) *given any neighborhood $V \subset X$ of $\widehat{X}$, every entire solution $\{\vartheta_i \in X_i\}$ is eventually contained in V, i.e., $\exists\, N$ such that for all $n \geq N$, $\vartheta_n \in V$;* (iii) *given any neighborhood $H \subset X \times X$ of $\widehat{F}$, and any entire solution $\{\vartheta_i \in X_i\}$, $\exists\, N$ such that for all $n \geq N$, $(v_n, v_{n+1}) \in H$;* (iv) *if $x \in X_n$, then $F_n(x)$ is nonempty.*

Proof. (i), (ii), and (iii) follow from the definition of lim sup of sets and noting that X and $X \times X$ are compact spaces. Statement (iv) follows from the definition of F_n. ∎

Having defined base relations from a process or NDEs, we study the dynamics of such closed relations. One can define invariant sets, attractors, etc., as in the case of maps. This is the content of the following subsection.

2.2. Asymptotic dynamics of closed relations. Researchers belonging to different disciplines have contributed to the theory of iteration of maps for more than a century. A far-reaching generalization of maps that has been considered is to iterate relations. Closed relations on a compact space are set-valued dynamical systems which are upper semicontinuous [1]. All the basics of topological dynamics (including definitions of limit sets and attractors) of closed relations are well developed in Akin's book [1]. McGehee [15] and McGehee and Wiandt [16] provide a lucid development of the same basic theory with different proofs. We recall some definitions from this literature which will be useful to us.

We know that a relation F on a set X is a subset of $X \times X$. The image of a point under a relation F is defined as $F(x) := \{y : (x, y) \in F\}$. More generally, if $S \subset X$,

$$F(S) := \{y : (x, y) \in F, \text{ where } x \in S\}.$$

Recall that a map is a relation F with the additional property that, for every $x \in X$, there exists a unique $y \in X$ satisfying $(x, y) \in F$. Since in general, $F(x)$ is not necessarily a singleton but a subset of X, a relation can be treated as a set-valued map. We point out that viewing relations as set-valued maps would not help us, hence we treat relations as relations.

To obtain dynamics from a map, we iterate the map by self-compositions. To iterate relations we have to compose them too, and a customary generalization of the definition of composition of maps is the following.

DEFINITION 2.8. *If F and G are relations on X, then the composition of F with G is the relation*

$$G \circ F \equiv \left\{ (x,z) \in X \times X : \exists\, y \text{ such that } (x,y) \in F \text{ and } (y,z) \in G \right\}.$$

The relation obtained by an n-fold composition of a relation F with itself is denoted by F^n. For a map, an orbit is the succession of images of a point. For a relation, a point may have none or many image points. An orbit for a relation is one of the possible successions of images.

DEFINITION 2.9. *A sequence $\{p_i\}_{i\in I}$ is said to be an* orbit *of F if $(p_i, p_{i+1}) \in F$ for all $i \in I$, where I is a finite or infinite interval of $\mathbb{Z}$ (i.e., $\{p_i\}$ could be either a left-(in)finite or a right-(in)finite or a bi-(in)finite sequence with every tuple $(p_i, p_{i+1}) \in F$).*

The analog of the inverse of a map is obtained for a relation by defining its transpose.

DEFINITION 2.10. *If F is a relation on X, then its transpose F^* is defined by $F^* := \{(y,x) : (x,y) \in F\}$.*

A closed relation on X is a closed subset of $X \times X$. Closed relations have useful properties: (i) the composition of two closed relations is closed; (ii) the transpose of a closed relation is closed. Henceforth we consider only closed relations. Next, we recall definitions pertinent to the asymptotic dynamics of closed relations. In particular, the attractor and attractor block definitions below are centrally relevant for later usage.

For a continuous map $h : X \to X$, we know that a set $A \subset X$ is invariant under h if $h(A) = A$. When X is compact, we know that for any set S the set $\omega(S;h) := \limsup_{n\to\infty} h^n(S)$ is invariant under h. However, if F is a relation on X and for $S \subset X$, the set $\limsup_{n\to\infty} F^n(S)$ is in general not invariant (see [15]) and the ω-limit of S under a relation is slightly more technical [15].

If F is a relation on X, then a set $K \subset X$ is called a *confining* set if $F(K) \subset K$. It is intuitive to imagine that the iterates of S under F eventually enter one or more confining sets when X is compact. We collect all such confining sets:

$$\mathfrak{K}(S\,;\,F) = \left\{ K : K \text{ is a closed confining set satisfying } F^n(S) \subset K \text{ for some } n \geq 0 \right\}.$$

The ω-limit set of a set S under the relation F is then defined to be

$$\omega(S\,;\,F) = \bigcap \mathfrak{K}(S\,;\,F).$$

It is verified in [15] that if F is a closed relation on a compact space X and if $S \subset X$, then $\omega(S\,;\,F)$ is a closed invariant set. Further, for a closed relation on a compact space X the ω-limit set thus defined always enjoys these properties (see [15]): (i) $\limsup_{n\to\infty} F^n(S) \subset \omega(S\,;\,F)$; (ii) $S' \subset S \Longrightarrow \omega(S'\,;\,F) \subset \omega(S\,;\,F)$; (iii) $H \subset F \Longrightarrow \omega(S\,;\,H) \subset \omega(S\,;\,H)$; (iv) $\omega(S\,;\,F) = \omega(F^n(S)\,;\,F)$; (v) if S is closed and $F(S) = S$, then $\omega(S\,;\,F) = S$.

We next recall the definition of attractors and repellers for relations. Closely related to the definition of an attractor is the definition of its dual repeller.

DEFINITION 2.11. *A set* $A \subset X$ *is called an* attractor *for a relation* F *on* X *if there exists a neighborhood* U *of* A *such that* $\omega(U\,;\,F) = A$. *A set* $R \subset X$ *is called a* repeller *of* F *if there exists a neighborhood* U *of* R *such that* $\omega(U\,;\,F^*) = R$.

Every attractor has an attracting region called the basin. The *basin* of A is defined as $\{x \in X : \omega(x\,;\,F) \subset A\}$. For a closed relation F on a compact X, the basin of an attractor A is always open in X and contains A. The set $A^* := (Basin(A))^c$ is called the *dual repeller* of A. It is verified in [1, 15] that A^* is indeed a repeller of F and hence the terminology "dual repeller." Further if the closed relation were to be the graph of a continuous map, then the attractor definition of the closed relation coincides with that of a map (see [15] for details).

Closely related to the definition of an attractor is that of an attractor block which is repeatedly recalled in this paper.

DEFINITION 2.12. *A set* B *is called an* attractor block *for* F *if* $F(\overline{\mathsf{B}}) \subset Int(\mathsf{B})$.

It may be noted that by the definition of an attractor block, it follows that if B is an attractor block of F, so is $\overline{\mathsf{B}}$. Also, $\omega(\mathsf{B}\,;\,F)$ has a simplification in Theorem 2.1 below. Theorem 2.1 is rephrased from [15, Theorem 7.2 and Corollary 7.5].

THEOREM 2.1 (see [15]). *If* F *is a closed relation on a compact space* X*, then every attractor block* B *contains an attractor* A *of* F*, which is given by*

$$A = \bigcap_{k=1}^{\infty} F^k(\mathsf{B}) = \omega(\mathsf{B}\,;\,F).$$

Conversely, if A *is any attractor of* F*, then every neighborhood of* A *in* X *contains an attractor block* B *such that* B *is also a neighborhood of* A *in* X*, and* $\omega(\mathsf{B}\,;\,F) = A$.

It follows from Theorem 2.1 that every closed relation on a compact space has at least one attractor. This follows from the fact that $F(X) \subset X = Int(X)$. Also, the intersection of any two attractors is another attractor [1]. The following fundamental result connects the limit sets of an orbit of F to an attractor A and its dual repeller.

THEOREM 2.2 (see [15]). *If* A *is an attractor and if* $\{p_i\}_{i\in\mathbb{Z}}$ *is an orbit for a closed relation on a compact space* X*, then at least one of the following holds:* (i) $\{p_i\} \subset A$ *or* $\{p_i\} \subset A^*$*;* (ii) $\omega(\{p_i\}) \subset A$ *and* $\alpha(\{p_i\}) \subset A^*$*, where* $\omega(\{p_i\}) := \bigcap_{n\geq 0} \overline{\{p_k : k \geq n\}}$ *and* $\alpha(\{p_i\}) := \bigcap_{n\leq 0} \overline{\{p_k : k \leq n\}}$.

An attractor A decomposes X into a disjoint union, $X = A \cup A^* \cup Basin(A) - A$. The set $Basin(A) - A$ is called the set of connecting orbits associated with A. The Conley decomposition theorem (see [16]) stated in Theorem 1.1 tells us that the chain-recurrent set $\mathfrak{CR}(F)$ (for the original definition and motivation of the chain-recurrent set see [1, 16]) is the union of all its connecting orbits.

PROPOSITION 2.3. *If an attractor* A *is not entirely contained in* $\mathfrak{CR}(F)$*, then there exists a nonempty attractor* $A_0 \subsetneq A$.

Proof. If $A \subsetneq \mathfrak{CR}(F)$, then by the Conley decomposition theorem, there exists a point $x \in A$ such that $x \in Basin(A_0) - A_0$ for some attractor A_0. Hence $\omega(x\,;\,F) \subset A_0$. Moreover, since $x \in A$, $\omega(x\,;\,F) \subset A$. Hence $A_0 \cap A \neq \emptyset$. Since the intersection of two attractors is also another attractor, we can suppose that $A_0 \subset A$. ◻

3. Uniform attractors for an NDE via closed relations. The aim of this section is to prove the existence of uniform attractors for an NDE as stated in Lemma 3.1 which will be later used in section 4. The premises that appear in Lemma 3.1 may seem to be artificial, but it will be seen in section 4 that these are satisfied to prove the main results of this paper.

Given a process ϕ on X and an association $\{X_n\}$, we know from Definition 2.7 that the base relation $\widehat{F}$ is a closed relation on the base domain $\widehat{X}$. Of course, $\widehat{F}$ is also a closed relation on X since $\widehat{X} \subset X$. But from now on, we treat $\widehat{F}$ as a closed relation on $\widehat{X}$ only. We endow $\widehat{X}$ with the subspace topology induced from the topology on X. Hence $(\widehat{X}, d)$ is a metric space, where d was a metric defined on X. With that understanding, when the relation $\widehat{F}$ is involved, the definition of closure or interior is in reference to the topology on $\widehat{X}$. Hence $Int(\widehat{X}) = \widehat{X}$, and the attractors and repellers of $\widehat{F}$ are closed subsets of the subspace $\widehat{X}$. Considering this subspace topology neither offers us any advantage in this paper in proving our results, nor complicates the proofs, but only keeps some of the intermediate lemmas valid for any association instead of a natural association alone.

Given a process ϕ on X, an association $\{X_n\}$, and its base relation $\widehat{F}$, we call an attractor of the base relation $\widehat{F}$ a *base attractor*. Similarly, we call a repeller of $\widehat{F}$ a *base repeller*. In the following, we denote the (Pompeiu–Hausdorff) Hausdorff metric on the subsets of X by $d_H(A, B)$, i.e., $d_H(A, B) := \max(dist(A, B), dist(B, A))$.

LEMMA 3.1. *Let ϕ be a process on X with the natural association $\{X_n\}$, its base domain $\widehat{X}$, and base relation $\widehat{F}$. Then, if*

(a) *the process ϕ has equicontinuity, i.e., if for $\epsilon > 0$ there exists a $\delta > 0$ independent of x and y such that $d(x, y) < \delta \Longrightarrow d(\phi(n + 1, n, x), \phi(n + 1, n, y)) < \epsilon$ for all n, and*

(b) *$X_n \subset \widehat{X}$ and $F_n \subset \widehat{F}$ for all n (where F_n is as defined in Definition 2.7), and*

(c) *for each X_n, there exists a subsequence $\{X_{j_k}\}$ of $\{X_n\}$ such that $d_H(X_{j_k}, X_n) \to 0$ as $k \to \infty$, and*

(d) *C is a nonempty attractor of $\widehat{F}$, and*

(e) *$A_n := \phi(n, n-1, C \cap X_{n-1})$,*

then the following hold:

(i) *$A_n \subset C$ and A_n is nonempty for all n;*

(ii) *$\{A_n\}$ is a local +invariant uniform attractor w.r.t. the association $\{X_n\}$, i.e., it is simultaneously both a local +invariant uniform pullback attractor and local +invariant uniform forward attractor;*

(iii) *given any entire solution $\{\vartheta_n\}$ then at least one of these holds:* (iiia) *$\{\vartheta_n\} \subset C$;* (iiib) *the ω-limit set of $\{\vartheta_n\} \subset C$ and the α-limit set of $\{\vartheta_n\} \subset C^*$;* (iiic) *$\{\vartheta_n\} \subset C^*$, where C^* is the dual repeller of C. In particular, if* (iiia) *or* (iiib) *holds, $\lim_{n\to\infty} dist(\vartheta_n, A_n) = 0$;*

(iv) *there exists a local pullback attractor $\{E_n\}$ w.r.t. $\{X_n\}$ such that $E_n \subset A_n$ for all n, and E_n is nonempty for all n.*

The following lemmas prepare for the proof of Lemma 3.1.

LEMMA 3.2. *Let ϕ be a process on a compact space X with an association $\{X_n\}$, its base domain $\widehat{X}$, and base relation $\widehat{F}$. Then* (i) *for every $x \in \widehat{X}$, $\widehat{F}(x)$ is nonempty;* (ii) *$\widehat{F}$ is surjective, i.e., for every $y \in \widehat{X}$, there exists an $x \in \widehat{X}$ such that $(x, y) \in \widehat{F}$;* (iii) *there exists an attractor of $\widehat{F}$ which contains the ω-limit set of any orbit $\{p_i\}$ of $\widehat{F}$.*

Proof. (i) Let $x \in \widehat{X}$. Since $\hat{X} = \limsup_{n\to\infty} X_n$, there exists $x_{n_k} \in X_{n_k}$ such that $x_{n_k} \to x$ as $k \to \infty$. Now consider a sequence $\{y_{n_k} : y_{n_k} \in F_{n_k}(x_{n_k})\}$, where F_n is as defined in Definition 2.7. Clearly, $(x_{n_k}, y_{n_k}) \in F_{n_k}$. Since $\widehat{F} = \limsup_{n\to\infty} F_n$ and $X \times X$ is a compact space, every accumulation point of $\{(x_{n_k}, y_{n_k})\}$ (as $k \to \infty$) is contained in $\widehat{F}$ and is of the form $(x, *)$, where $*$ is an accumulation point of $\{y_{n_k}\}$. As $* \in \widehat{F}(x)$, $\widehat{F}(x)$ is nonempty.

(ii) Let $y \in \widehat{X}$. Then there exists $y_{n_k} \in X_{n_k}$ such that $y_{n_k} \to y$ as $k \to \infty$. Now, if $y_{n_k} \in X_{n_k}$, there exists $x_{n_k} \in X_{n_k-1}$, such that $(x_{n_k}, y_{n_k}) \in F_{n_k-1}$. Since $\widehat{F} = \limsup_{n\to\infty} F_n$ and $X \times X$ is compact, we have $(*, y) \in \widehat{F}$, where $*$ is an accumulation point of $\{x_{n_k}\}$. Such a $*$ belongs to $\widehat{X}$ since $\widehat{F}$ is a relation on $\widehat{X}$.

(iii) Statement (iii) follows from the fact that $\widehat{X}$ is an attractor block and $\omega(\widehat{X}, \widehat{F}) = \widehat{X}$ since $\widehat{F}$ is surjective. By Theorem 2.1, $\widehat{X}$ is an attractor, and as a trivial consequence contains the ω-limit set of every orbit of $\widehat{F}$. ∎

LEMMA 3.3. *Let ϕ be a process on X, $\{X_n\}$ its natural association with its base domain $\widehat{X}$ and base relation $\widehat{F}$. Suppose that $X_n \subset \widehat{X}$ and $F_n \subset \widehat{F}$ for all n, where F_n is as defined in Definition* 2.7. *Then every entire solution of ϕ is an orbit of $\widehat{F}$. In general, for any set $E \subset X_n$, $\phi(n+j, n, E) \subset \widehat{F}^{j-1}(E)$ for all $j > 1$.*

Proof. Let $\{\vartheta_i\}$ be an entire solution. By statement (iii) of Lemma 2.2 we know that $\vartheta_i \in X_i$ for all i. Since $\vartheta_{n+1} = \phi(n+1, n, \vartheta_n)$, by definition of F_n, $F_n(\vartheta_n) = \vartheta_{n+1}$. Hence $(\vartheta_n, \vartheta_{n+1}) \in F_n$. Since $F_n \subset \widehat{F}$, we have $(\vartheta_n, \vartheta_{n+1}) \in \widehat{F}$. This implies that $\{\vartheta_i\}$ is an orbit of $\widehat{F}$. By a similar argument, if $E \subset X_n$, $\phi(n+1, n, E) \subset F_n(E) \subset \widehat{F}(E)$ which implies $\phi(n+j, n, E) \subset \widehat{F}^{j-1}(E)$ for all $j > 1$. ∎

LEMMA 3.4. *Let ϕ be a process on X, and let $\{X_n\}$ be its natural association, $\widehat{X}$ its base domain, and $\widehat{F}$ its base relation w.r.t. $\{X_n\}$. Let C be a nonempty attractor of $\widehat{F}$. Suppose that for each X_n, there exists a subsequence $\{X_{j_k}\}$ of $\{X_n\}$ such that $d_H(X_{j_k}, X_n) \to 0$ as $k \to \infty$. Then $X_n \cap C \neq \emptyset$ for all n.*

Proof. Suppose for some n_0 it holds that $X_{n_0} \cap C = \emptyset$. Noting $C \subset \widehat{X}$, by the definition of $\widehat{X}$, for each $x \in C$, there exists a sequence $x_{n_k} \in X_{n_k}$ such that $x_{n_k} \to x$ as $k \to \infty$. Hence every neighborhood of $x \in C$ has a nonempty intersection with some X_n. In particular, let B be an attractor block such that $\omega(\mathsf{B}\ ;\ \widehat{F}) = C$. Since B contains a neighborhood of C (by Theorem 2.1), $X_n \cap \mathsf{B} \neq \emptyset$ for some n. Since $X_{n+j} \subset \widehat{F}^j(X_n)$ by Lemma 3.3, and by definition of an attractor block, $X_{n+j} \cap \mathsf{B} \neq \emptyset$ for all $j \geq 1$. Suppose that $X_n \cap \mathsf{B} \neq \emptyset$, $X_n \cap C = \emptyset$, and $dist(C, X_n \cap \mathsf{B}) = 2\epsilon > 0$. Let B' be another attractor block such that $\omega(\mathsf{B}'\ ;\ \widehat{F}) = C$ such that $dist(C, \mathsf{B}') < \epsilon$. This implies $dist(X_n, \mathsf{B}') > \epsilon$ and hence $d_H(X_n, \mathsf{B}') > \epsilon$. As a consequence of Theorem 2.1, there exists an integer K such that for all $k \geq K$, $X_{n+k} \cap \mathsf{B}' \neq \emptyset$. This implies $d_H(X_{n+k}, X_n) > \epsilon$ for all $k \geq K$ since we had $d_H(X_n, \mathsf{B}') > \epsilon$. This contradicts the assumption that there exists a subsequence $\{X_{j_k}\}$ of $\{X_n\}$ such that $d_H(X_{j_k}, X_n) \to 0$ as $k \to \infty$. Hence $X_n \cap C \neq \emptyset$ and thus by definition of C and Lemma 3.3, $X_{n+j} \cap C \neq \emptyset$. Since $\exists\ \{X_{i_k}\}$ such that $d_H(X_{i_k}, X_{n_0}) \to 0$ as $k \to \infty$, it follows that $X_{n_0} \cap C \neq \emptyset$. ∎

Proof of Lemma 3.1. Throughout, for the sake of brevity, when we mention a ϕ-invariant or a ϕ +invariant attractor, it is understood that it is so always w.r.t. to the natural association $\{X_n\}$.

(i) Suppose C is a nonempty attractor of $\widehat{F}$. From Lemma 3.3, for any n, $\phi(n, n-1, C \cap X_{n-1}) \subset C$. Hence $A_n \subset C$ for all n. Suppose $A_{n_0} = \{\emptyset\}$ for some n_0. The definition of $A_n \Rightarrow C \cap X_{n_0-1} = \emptyset$. This contradicts Lemma 3.4. Hence each A_n is nonempty.

(ii) We now show that $A_n := \phi(n, n-1, C \cap X_{n-1})$ is a +invariant uniform local pullback attractor assuming that ϕ has equicontinuity.

By the definition of an attractor block, and from Theorem 2.1 we can choose an attractor block B of $\widehat{F}$ with the following properties: $\omega(\mathsf{B}; \widehat{F}) = \bigcap_{j=1}^{\infty} \widehat{F}^j(\mathsf{B}) = C$; B is closed in $\widehat{X}$, and we can choose $\eta > 0$ such that B contains $B_\eta(C) \cap \widehat{X}$. Define the set B_n, a closed subset of B, by $\mathsf{B}_n := \mathsf{B} \cap X_n$.

Since $B_\eta(C) \subset \mathsf{B}$ and $A_n \subset C$ for all n (by Lemma 3.3), we have

$$B_\eta^{(n)}(A_n) \subset \mathsf{B} \cap X_n = \mathsf{B}_n \text{ for all } n. \tag{10}$$

Next, noting that $\phi(n, n-1, C) \subset C$ (from Lemma 3.3) and $\phi(n, n-1, X_{n-1}) = X_n$ we check that $\{A_n\}$ is ϕ +invariant by

$$\begin{aligned}\phi(n+1, n, A_n) &= \phi(n+1, n, \phi(n, n-1, C \cap X_{n-1}))\\ &\subset \phi(n+1, n, C \cap X_n)\\ &= A_{n+1}.\end{aligned}$$

Next, we verify the uniform attractive property. Fix $\epsilon > 0$. Since ϕ has equicontinuity, there exists a $\delta > 0$ such that

$$dist(\phi(n+1, n, B_\delta^{(n)}(E)) \ , \ \phi(n+1, n, E)) < \epsilon \tag{11}$$

for every nonempty set $E \subset X_n$ and for all n.

Let C_δ denote $B_\delta(C)$, i.e., $C_\delta := \{x \in \widehat{X} : d(x, C) < \delta\}$. Since $\omega(\mathsf{B}; \widehat{F}) = \bigcap_{j=1}^\infty \widehat{F}^j(\mathsf{B}) = C$, there exists an integer J such that $\widehat{F}^j(B) \subset C_\delta$ for all $j > J$. Now if $j-1 \geq J$, for any n,

$$\phi(n-1, n-j, \mathsf{B}_{n-j}) \overset{\text{Lemma 3.3}}{\subset} \widehat{F}^{j-1}(\mathsf{B}) \ \subset \ C_\delta.$$

Also since $\phi(n-1, n-j, \mathsf{B}_{n-j}) \subset X_{n-1}$, we have $\phi(n-1, n-j, \mathsf{B}_{n-j}) \subset C_\delta \cap X_{n-1}$. Using this, we get

$$\begin{aligned}\phi(n, n-j, \mathsf{B}_{n-j}) &= \phi(n, n-1, \phi(n-1, n-j, \mathsf{B}_{n-j}))\\ &\subset \phi(n, n-1, C_\delta \cap X_{n-1}). \qquad (12)\end{aligned}$$

To prove that $\{A_n\}$ is a +invariant uniform local pullback attractor, it is sufficient to show that for all $j \geq J$, $\sup_n dist(\phi(n, n-j, B_\eta^{(n-j)}(A_{n-j})), A_n) \leq \epsilon$. Let $j \geq J$. Then for any n,

$$\begin{aligned}dist(\phi(n, n-j, \mathsf{B}_{n-j}), A_n) &\leq dist(\phi(n, n-1, C_\delta \cap X_{n-1}), A_n) \ \ \text{(due to (12))} \qquad (13)\\ &= dist(\phi(n, n-1, C_\delta \cap X_{n-1}), \phi(n, n-1, C \cap X_{n-1})) \qquad (14)\\ &< \epsilon \quad \text{(due to (11))}. \qquad (15)\end{aligned}$$

Since by (10), $B_\eta^{(n-j)}(A_{n-j}) \subset \mathsf{B}_{n-j}$ for all $j \geq J$, it holds that

$$\sup_n dist(\phi(n, n-j, B_\eta^{(n-j)}(A_{n-j}), A_n)) \leq \epsilon.$$

The proof that $\{A_n\}$ is a +invariant uniform local forward attractor is similar: by replacing n by $n+j$ in (13), and repeating the steps (13)–(15) we get for all $j \geq J$,

$$\sup_n \ dist(\phi(n+j, n, B_\eta^{(n)}(A_n)) \ , \ A_{n+j}) \leq \epsilon.$$

(iii) By Lemma 3.3, $\{\vartheta_n\}$ is an orbit of $\widehat{F}$ as well. Applying Theorem 2.2 we have (iiia), (iiib), and (iiic) of Lemma 3.1. Suppose (iiia) or (iiib) holds. We shall show that $dist(\vartheta_n, A_n) \to 0$ as $n \to \infty$. If (iiia) or (iiib) holds, then for all n sufficiently large, $\vartheta_n \subset \mathsf{B}_n$ since B_n contains $B_\eta(C) \cap X_n$. Replacing n by $n+j$ in (13)–(15) we arrive at

$$\lim_{j \to \infty} dist(\phi(n+j, n, \mathsf{B}_n), A_{n+j}) = 0$$

for any n. Since $\vartheta_n \subset \mathsf{B}_n$, it directly follows that $\lim_{j\to\infty} dist(\vartheta_{n+j}, A_{n+j}) = 0$.

(iv) Define

$$E_n := \bigcap_{j=1}^{\infty} \phi(n, n-j, \mathsf{B}_{n-j}), \tag{16}$$

where B_{n-j} is is as above in the proof of (ii). We now claim that $\{E_n\}$ is a local pullback attractor such that $E_n \subset A_n$.

We first show that $\{\mathsf{B}_n\}$ is a ϕ +invariant set: $\phi(n+1, n, \mathsf{B}_n) \subset \widehat{F}(\mathsf{B}_n) \subset Int(\mathsf{B})$. But we also have $\phi(n+1, n, \mathsf{B}_n) \subset X_{n+1}$. Hence $\phi(n+1, n, \mathsf{B}_n) \subset Int(\mathsf{B}) \cap X_{n+1} \subset \mathsf{B} \cap X_{n+1} = \mathsf{B}_{n+1}$.

Since $\{\mathsf{B}_n\}$ is ϕ +invariant, we have $\phi(n, n-j, \mathsf{B}_{n-j}) \supset \phi(n, n-j-1, \mathsf{B}_{n-j-1})$. Hence $\bigcap_{j=1}^{\infty} \phi(n, n-j, \mathsf{B}_{n-j})$ is a nested intersection of closed subsets. Since ϕ is continuous, applying (4), we have $\phi(n+1, n, E_n) = E_{n+1}$. Hence $\{E_{n+1}\}$ is ϕ-invariant.

Next we show that $E_n \subset A_n$. First we observe $E_n \subset C$ since

$$E_n = \bigcap_{j=1}^{\infty} \phi(n, n-j, \mathsf{B}_{n-j}) \overset{\text{Lemma 3.3}}{\subset} \bigcap_{j=1}^{\infty} \widehat{F}^j(\mathsf{B}) = C.$$

Now, $E_n = \phi(n, n-1, E_{n-1}) = \phi(n, n-1, C \cap E_{n-1}) \subset \phi(n, n-1, C \cap X_{n-1}) = A_n$. Thus $E_n \subset A_n$.

Since E_n is obtained as an intersection of $\phi(n, n-j, \mathsf{B}_{n-j})$ over all $j \geq 1$, by definition of *dist* we have

$$\lim_{j\to\infty} dist(\phi(n, n-j, \mathsf{B}_{n-j}), E_n) = 0.$$

Also, in (10), we had $\mathsf{B}_{n-j} \supset B_\eta^{(n-j)}(A_{n-j})$, and thus $\mathsf{B}_{n-j} \supset B_\eta^{(n-j)}(E_{n-j})$. Using this we get

$$\lim_{j\to\infty} dist(\phi(n, n-j, B_\eta^{(n-j)}(E_{n-j})), E_n) = 0.$$

This proves $\{E_n\}$ is a local pullback attractor.

Since $A_n \subset \mathsf{B}_n$, each B_n is also nonempty. Thus $\bigcap_{j=1}^{\infty} \phi(n, n-j, \mathsf{B}_{n-j})$ is a nested intersection of nonempty closed subsets, and since X is compact, E_n is nonempty for all n. ∎

4. Dynamical decomposition: Stationary input. In this section, we state the first of the two main results of the paper in Theorem 4.1—this is a reformulation of the results (i)–(iv) of Theorem 1.2. Lemmas 3.1 and 4.4 guide the proof of Theorem 4.1.

THEOREM 4.1. *Let U be a complete metric space and $\xi_n : \Omega \to U$ a stationary process defined on a probability space $(\Omega, \mathcal{F}, P)$. Also, let $g : U \times X \to X$ be any uniformly continuous map, where X is a compact metric space and let g define an RDE as in* (1)*. Consider the set of all NDE on X obtained by $g_{n,\omega}(\cdot) := g(\xi_n(\omega), \cdot)$, and their corresponding processes ϕ_ω, where $\omega \in \Omega$. For the process of $\{g_{n,\omega}\}$, let $\{X_n(\omega)\}$ be its natural association, $\widehat{X}(\omega)$ its base domain, $\widehat{F}(\omega)$ its base relation, and $C(\omega)$ any nonempty base attractor, i.e., an attractor of $\widehat{F}(\omega)$. Then there exists a set Ω_0 of probability 1 such that for all $\omega \in \Omega_0$,*

(i) *every entire solution of ϕ_ω is an orbit of $\widehat{F}(\omega)$;*

(ii) *$A_n(\omega) := \phi_\omega(n, n-1, C(\omega) \cap X_{n-1}(\omega))$ is a local +invariant uniform attractor of ϕ_ω w.r.t. $\{X_n(\omega)\}$ such that $A_n(\omega) \neq \emptyset$ for all n;*

(iii) *given any entire solution* $\{\vartheta_n\}$ *of* $\{g_{n,\omega}\}$ *and the base attractor* $C(\omega)$*, then one of these hold:* (iiia) $\{\vartheta_n\} \subset C(\omega)$*;* (iiib) *the* ω*-limit set of* $\{\vartheta_n\} \subset C(\omega)$ *and the* α*-limit set of* $\{\vartheta_n\} \subset C^*(\omega)$*;* (iiic) $\{\vartheta_n\} \subset C^*(\omega)$*, where* $C^*(\omega)$ *is the dual repeller of* $C(\omega)$*. In particular, whenever* (iiia) *or* (iiib) *holds,* $\lim_{n\to\infty} dist(\vartheta_n, A_n(\omega)) = 0$*;*

(iv) *there also exists a local pullback attractor* $\{E_n(\omega)\}$ *w.r.t.* $\{X_n(\omega)\}$ *such that* $\emptyset \neq E_n(\omega) \subset A_n(\omega)$ *for all* n*;*

(v) *if* $C(\omega)$ *is not contained entirely in* $\mathfrak{CR}(\widehat{F}(\omega))$*, the chain-recurrent set of* $\widehat{F}(\omega)$*, then there exists at least one nonempty subattractor* $C_0(\omega) \subsetneq C(\omega)$ *of* $\widehat{F}(\omega)$*.*

In proving Theorem 4.1 we also lean on methods from ergodic theory (e.g., [11, 19]). We recall some relevant definitions.

A metric dynamical system is a quadruplet $(\Omega, \mathcal{F}, \mu, T)$, where $(\Omega, \mathcal{F}, \mu)$ is a measure space and $T : \Omega \to \Omega$ is a measurable map. A metric dynamical is said to be a measure preserving dynamical system (MPDS) if $\mu(T^{-1}(A)) = \mu(A)$ for all A in $\mathcal{F}$. An MPDS $(\Omega, \mathcal{F}, \mu, T)$ is said to be ergodic if $T^{-1}(A) = A$ for any $A \in \mathcal{F}$ implies $\mu(A) = 0$ or $\mu(A) = 1$. An attribute of an MPDS is the notion of qualitative recurrence. A point $x \in \Omega$ is said to be recurrent if there exists a subsequence $\{n_k\}$ of natural numbers such that $T^{n_k}(x) \to x$ as $k \to \infty$. We recall Poincaré's recurrence theorem in a modern formulation (e.g., [9]) to essentially point out that the set of all recurrent points has full measure when the space is complete and separable:

THEOREM 4.2. *Let* $(\Omega, \mathcal{F}, \mu, T)$ *be an MPDS, where* $\mu(\Omega) < \infty$*. Let* $E \in \mathcal{F}$ *and* $\mu(E) > 0$*. Then almost all points of* E *return infinitely often to* E *under positive iteration of* T*, i.e., there exists a set* $E' \subset E$ *such that* $\mu(E') = \mu(E)$ *and for all* $x \in E$ *there exist integers* $0 < n_1 < n_2 < \cdots$ *such that* $T^{n_k}(x) \in E'$ *for all* k*. Also, if* Ω *is a complete separable metric space with metric* d*, we have*

$$\mu\left(\left\{x \in E : \liminf_{n\to\infty} d(T^n(x), x) = 0\right\}\right) = \mu(E).$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space and S a separable complete metric space. Let $\mathcal{B}_S$ denote the Borel sigma-field of S. Let $\{\theta_n\}$ be an S-valued stationary process. Consider $(S^\infty, \mathcal{B}^\infty)$, where S^∞ is the Cartesian product of a bi-infinite countable number of copies of S and $\mathcal{B}^\infty$ the sigma-field generated by the product topology on S^∞. For each $\omega \in \Omega$, there exists a $\bar{u} = (\ldots, u_{-1}, u_0, u_1, \ldots) \in S^\infty$ such that $u_k = \theta_k(\omega)$. The process $\theta = \{\theta_n\}$ and P induce a measure μ on $(S^\infty, \mathcal{B}^\infty)$ defined by $\mu(A) := P(\theta^{-1}(A))$ for all $A \in \mathcal{B}^\infty$. Moreover, by definition of μ it follows that $\mu\{\bar{u} : \exists\, \omega \in \Omega \text{ such that } u_k = \theta_k(\omega) \text{ for all } k\} = 1$. It is well known (e.g., [11]) that $\{\theta_n\}$ is stationary if and only if $(S^\infty, \mathcal{B}^\infty, \mu, \sigma)$ is an MPDS where σ is the map that sends a point $\bar{u} := (\ldots, u_{n-1}, u_n, u_{n+1}, \ldots)$ in S^∞ to $\sigma(\bar{u}) = (\ldots, u_n, u_{n+1}, u_{n+2}, \ldots)$. In words, the map σ takes a sequence and shifts every symbol in the sequence one slot to the left. The map σ is known as the bilateral shift since σ is a bijection. Also, $\{\theta_n\}$ is ergodic if and only if the MPDS $(S^\infty, \mathcal{B}^\infty, \mu, \sigma)$ is ergodic (e.g., [11]).

In proving Theorem 4.1, we deal with stochastic processes which are set valued. To help us do that we set up some notation. When X is a metric space, we denote H_X and $\mathsf{H}_{X\times X}$ to be the collection of all nonempty closed subsets of X and $X \times X$, respectively. It is well known that whenever X is complete (compact), H_X is also a complete (compact) metric space under the Hausdorff metric $d_H(A, B)$. Also, we make use of the equivalence $d_H(A, B) = \max(dist(A, B), dist(B, A)) = \inf\{\epsilon : A \subset B_\epsilon(B) \text{ and } B \subset B_\epsilon(A)\}$.

Consider $S^{-\infty}$ to denote the Cartesian product of the left-infinite countable number copies of S and which will be equipped with the product topology. Then the following simple result in Lemma 4.1 is a standard result (e.g., [11]).

LEMMA 4.1. *Let $\{\theta_n\}_{n\in\mathbb{Z}}$ be an S-valued stationary process with an MPDS $(S^\infty, \mathcal{B}^\infty, \mu, \sigma)$. Then*

(i) *$\{\theta_{-n}\}$ is also stationary, and if $\{\theta_n\}$ is ergodic, then so is $\{\theta_{-n}\}$;*

(ii) *if R is some measurable space, and*

$$\Theta_i(\omega) := \Phi(\ldots, \xi_{i-2}(\omega), \xi_{i-1}(\omega)),$$

where $\Phi : S^{-\infty} \to R$ is a measurable function, then $\{\Theta_n\}$ is an R-valued stationary process. Further if $\{\theta_n\}$ is ergodic, then so is $\{\Theta_n\}$.

LEMMA 4.2. *Given $g : U \times X$, let X_n and ϕ_ω be functions of the U-valued stochastic process $\{\xi_n\}$ as specified in Theorem 4.1. Assuming Borel-sigma algebras defined on H_X and $\mathsf{H}_{X\times X}$ obtained by the Hausdorff distance, there exist measurable functions $\lambda : U^{-\infty} \to \mathsf{H}_X$ and $\Lambda : U^{-\infty} \to \mathsf{H}_{X\times X}$ such that for all n, $X_n(\omega) = \lambda(\ldots, \xi_{n-2}(\omega), \xi_{n-1}(\omega))$ and $F_n(\omega) = \Lambda(\ldots, \xi_{n-2}(\omega), \xi_{n-1}(\omega))$, where $F_n(\omega) = \{(x, \phi_\omega(n+1, n, x) : x \in X_n(\omega)\}$ is the relation on $X_n(\omega)$.*

Proof. We will exhibit the existence of a measurable function $\lambda : U^{-\infty} \to \mathsf{H}_X$ such that $X_n(\omega) = \lambda(\ldots, \xi_{n-2}(\omega), \xi_{n-1}(\omega))$. The proof of the existence of a measurable $\Lambda : U^{-\infty} \to \mathsf{H}_{X\times X}$ such that $F_n = \Lambda(\ldots, \xi_{n-2}(\omega), \xi_{n-1}(\omega))$ is similar.

For the U-valued stationary process $\{\xi_n\}$, let its MPDS be denoted by $(U^\infty, \mathcal{B}^\infty, \mu, \sigma)$.

Given any pair $i, n \in \mathbb{Z}$ such that $n > i$, we define $h_{n,i} : U^\infty \to \mathsf{H}_X$ by

$$h_{n,i}(\bar{u}) := g_{n-1} \circ \cdots \circ g_{i+1} \circ g_i(X),$$

where $\bar{u} = (\ldots, u_{-1}, u_0, u_1, \ldots) \in U^\infty$, $g_m : X \to X$ is defined by $g_m := g(u_m, \cdot)$, the map $g : U \times X \to X$ being as in Theorem 4.1.

Let d_U denote some metric on U that gives rise to $\mathcal{B}$. Then $d'_U := \min(1, d_U)$ also generates $\mathcal{B}$. Let $d_\infty(\bar{u}, \bar{v}) := \sum_{i=-\infty}^{\infty} \frac{d'_U(u_i, v_i)}{2^{|i|+1}}$ be the metric on U^∞. It may be verified that d_∞ generates the product topology on U^∞. We now claim that $h_{n,i} : U^\infty \to \mathsf{H}_X$ is a continuous map for each n and i. To show this let $\{\bar{u}_k\}$ be any sequence such that $\bar{u}_k \to \bar{u}$ as $k \to \infty$. We will show that $h_{n,i}$ is continuous by proving $h_{n,i}(\bar{u}_k) \to h_{n,i}(\bar{u})$ as $k \to \infty$.

Let $\bar{u}_k = (\ldots, u^k_{-1}, u^k_0, u^k_1, \ldots)$. Hence $h_{n,i}(\bar{u}_k) = g^{[k]}_{n-1} \circ \cdots \circ g^{[k]}_{k+1} \circ g^{[k]}_i(X)$, where $g^{[k]}_m := g(u^k_m, \cdot)$. Since $g : U \times X \to X$ is continuous, it follows from the continuity argument that given any ϵ there exists a $\delta > 0$ such that

$$d_U(u_m, u^k_m) < \delta \text{ for all } n \le m \le i \Rightarrow d\big(g_{n-1} \circ \cdots \circ g_i(x)\ ,\ g^{[k]}_{n-1} \circ \cdots \circ g^{[k]}_i(x)\big) < \epsilon$$

$$\Rightarrow h_{n,i}(\bar{u}_k) \subset B_\epsilon(h_{n,i}(\bar{u})) \text{ and } h_{n,i}(\bar{u}) \subset B_\epsilon(h_{n,i}(\bar{u}_k)) \tag{17}$$

$$\Rightarrow d_H(h_{n,i}(\bar{u}), h_{n,i}(\bar{u}_k)) < \epsilon. \tag{18}$$

Since $\bar{u}_k \to \bar{u}$ as $k \to \infty$, we can find an integer K such that for all $k \ge K$, $d_\infty(\bar{u}, \bar{u}_k) < \frac{\delta}{2^{n-i}}$ holds. This implies $d_U(u_m, u^k_m) < \delta$ for all $n \le m \le i$. Hence for all $k \ge K$ from (18), we have $d_H(h_{n,i}(\bar{u}), h_{n,i}(\bar{u}_k)) < \epsilon$. Since ϵ was chosen arbitrarily, $h_{n,i}(\bar{u}_k) \to h_{n,i}(\bar{u})$ as $k \to \infty$. This implies that $h_{n,i}$ is continuous.

Define $h_n : U^\infty \to \mathsf{H}_X$ by

$$h_n(\bar{u}) := \bigcap_{j=1}^{\infty} h_{n,n-j}(\bar{u}). \tag{19}$$

Since $h_{n,n-j}$ is continuous for any $j \geq 1$, $h^{-1}_{n,n-j}(B) \in \mathcal{B}^\infty$ for any Borel subset B contained in H_X. This implies $h_n^{-1}(B) \in \mathcal{B}^\infty$ for any Borel subset B contained in H_X. This implies h_n is measurable.

For each $\omega \in \Omega$, there exists a $\bar{u}^\omega$ such that $u_k^\omega = \xi_k(\omega)$. Hence

$$\begin{aligned} X_n(\omega) &\overset{\text{by definition}}{=} \bigcap_{j=1}^{\infty} \phi_\omega(n, n-j, X) \\ &= \bigcap_{j=1}^{\infty} g(\xi_{n-1}, \cdot) \circ \cdots \circ g(\xi_{n-j+1}, \cdot) \circ g(\xi_{n-j}, X) \\ &= \bigcap_{j=1}^{\infty} h_{n,n-j}(\bar{u}^\omega) \\ &= h_n(\bar{u}^\omega). \end{aligned}$$

By the above deduction, it is also clear that $h_n(\bar{u}^\omega)$ is actually a function of only $(\ldots, \xi_{n-2}(\omega), \xi_{n-1}(\omega))$ and does not depend on $(\xi_n(\omega), \xi_{n+1}(\omega), \ldots)$. Hence there exists a function $\lambda : U^{-\infty} \to \mathsf{H}_X$ such that $\lambda(\ldots, \xi_{n-2}(\omega), \xi_{n-1}(\omega)) = h_n(\bar{u}^\omega)$. Since h_n is measurable, so is λ. ∎

COROLLARY 4.1. *Let the random variables $\{X_n\}$ and $\{F_n\}$ be defined as in Lemma* 4.2. *Then $\{X_n\}$ and $\{F_n\}$ are stationary (ergodic) if $\{\xi_n\}$ is stationary (ergodic).*

Proof. Using the fact that a measurable function λ exists such that $X_n(\omega) = \lambda(\ldots, \xi_{n-2}(\omega), \xi_{n-1}(\omega))$ and applying statement (ii) of Lemma 4.1, we get $\{X_n\}$ is stationary (ergodic) whenever $\{\xi_n\}$ is stationary (ergodic). By similar arguments, if $\{\xi_n\}$ is stationary (ergodic), then $\{F_n\}$ is stationary (ergodic). ∎

LEMMA 4.3. *Let $\{\theta_n\}$ be a stationary process defined on $(\Omega, \mathcal{F}, P)$ taking values in a separable complete metric space (S, d_S). Then for any $k \in \mathbb{Z}$, the following hold:*

$$P\left(\left\{\omega : \overline{\bigcup_{i=k}^{\infty} \theta_i(\omega)} = \overline{\bigcup_{i=k+1}^{\infty} \theta_i(\omega)}\right\}\right) = 1; \tag{20}$$

$$P\left(\left\{\omega : \liminf_{n\to\infty} d_S(\theta_k(\omega)\, ,\, \theta_{k+n}(\omega)) = 0\right\}\right) = 1. \tag{21}$$

Proof. Suppose that for some k and ω it holds that $\theta_k(\omega) \subset \bigcap_{i=k+1}^{\infty} \overline{\bigcup_{n\geq i} \theta_n(\omega)}$. This implies $\theta_k(\omega) \subset \overline{\bigcup_{i=k+1}^{\infty} \theta_i(\omega)} \Rightarrow \overline{\bigcup_{i=k}^{\infty} \theta_i(\omega)} = \overline{\bigcup_{i=k+1}^{\infty} \theta_i(\omega)}$.

Let $(n_j \uparrow)$ denote a strictly increasing subsequence of natural numbers; also for shortness let the symbols "s.t." stand for the phrase "such that." Now, $\theta_k(\omega) \subset \bigcap_{i=k+1}^{\infty} \overline{\bigcup_{n\geq i} \theta_n} \Leftrightarrow$ there exists $(n_j \uparrow)$ such that $\lim_{j\to\infty} d(\theta_k(\omega), \theta_{k+n_j}(\omega)) = 0$. So to prove (20) it is enough to show

$$P\left(\left\{\omega : (n_j(\omega) \uparrow) \ \text{ s.t. } \ \lim_{j\to\infty} d(\theta_k(\omega), \theta_{k+n_j}(\omega)) = 0\right\}\right) = 1. \tag{22}$$

Let $(S^\infty, \mathcal{B}^\infty, \mu, \sigma)$ be the MPDS corresponding to the process $\{\theta_n\}$. If $d'_S := \min(1, d_S)$, then it may be verified that $d_\infty(\bar{u}, \bar{v}) := \sum_{i=-\infty}^{\infty} \frac{d'_S(u_i, v_i)}{2^{|i|+1}}$ is a metric on S^∞ such that S^∞ is a separable complete metric space (since S is separable complete). To prevent cluttering of notation, denote $E := S^\infty$. By application of Poincaré's recurrence theorem (Theorem 4.2), we have

$$\mu\Big(\big\{\bar{u} \in E : \liminf_{n\to\infty} d_\infty(\sigma^n(\bar{u}), \bar{u}) = 0\big\}\Big) = \mu(E) = 1. \tag{23}$$

This implies for almost all $\bar{u} \in E$ w.r.t. μ, there exists a subsequence $(n_j(\bar{u})\uparrow)$ such that $\sigma^{n_j}(\bar{u})$ is a Cauchy sequence and converges to $\bar{u}$. By the definition of convergence in E, it follows that $\{u_{k+n_j}\}$ is also a Cauchy sequence and converges to u_k. Hence,

$$\mu\Big(\{\bar{u} \in E : \exists\,(n_j(\bar{u})\uparrow) \;\text{ s.t. }\; \lim_{j\to\infty} d_S(u_{k+n_j}, u_k) = 0\}\Big) = \mu(E) = 1. \tag{24}$$

Note that for almost all $\bar{u} \in E$ w.r.t. μ, there exists a sequence $\{\theta_n(\omega)\}$ such that $u_n = \theta_n(\omega)$ for all n. Also, since by definition $\mu(E) = P((\ldots,\theta_{-1},\theta_0,\theta_1,\ldots)^{-1}(E))$, (24) implies

$$P\Big(\{\omega : \exists(n_j(\omega)\uparrow) \;\text{ s.t. }\; \lim_{j\to\infty} d_S(\theta_{k+n_j}(\omega), \theta_k(\omega)) = 0\}\Big) = 1. \tag{25}$$

This proves (22). Also, from (24), (21) follows. ∎

Lemma 4.4. *Let* $X_n(\omega)$, $\hat{X}(\omega)$, $F_n(\omega)$, *and* $\widehat{F}(\omega)$ *be as in Theorem* 4.1. *Then*

(i) $X_n(\omega) \subset \widehat{X}(\omega)$ *for all* n *with probability* 1*;*

(ii) $F_n(\omega) \subset \widehat{F}(\omega)$ *for all* n *with probability* 1.

Proof. We know $\{X_i\}$ is a stationary process taking values in H_X from Corollary 4.1. The space H_X is compact since X is compact and hence is complete and separable. From Lemma 4.3, we have

$$P\left(\left\{\omega : \overline{\bigcup_{i=n}^{\infty} X_i(\omega)} = \overline{\bigcup_{i=j}^{\infty} X_i(\omega)}\right\}\right) = 1$$

for any $n, j \in \mathbb{Z}$. By definition of lim sup, we have $\overline{\bigcup_{i\geq j} X_i(\omega)} = \bigcup_{i\geq j} X_i(\omega) \cup \limsup_{j\to\infty} X_j(\omega)$. Using this and $\limsup_{j\to\infty} X_j(\omega) = \widehat{X}(\omega)$ in the above expression,

$$P\left(\left\{\omega : \overline{\bigcup_{i=n}^{\infty} X_i(\omega)} = \bigcup_{i=j}^{\infty} X_i(\omega) \cup \widehat{X}(\omega)\right\}\right) = 1$$

for any $n, j \in \mathbb{Z}$. Hence for any given n, and all j,

$$P\left(\left\{\omega : X_n(\omega) \subset \bigcup_{i=j}^{\infty} X_i(\omega) \cup \widehat{X}(\omega)\right\}\right) = 1. \tag{26}$$

Now to arrive at (i) we use the fact that (26) holds for any j. Fix any ω where the set inclusion in (26) holds. Suppose there exists an integer J such that $X_n(\omega) \neq X_i(\omega)$ for all $i \geq J$. By setting $j = J$ in (26), if the set inclusion in (26) were to hold for any ω, then one should have $X_n(\omega) \subset \widehat{X}(\omega)$. Suppose if there exists no such J, then clearly $X_n(\omega)$ repeats infinitely often in the sequence $\{X_i(\omega)\}_{i\geq 0}$. This then would imply that $X_n(\omega) \subset \widehat{X}(\omega)$. From these arguments (26) implies $P(\{\omega : X_n(\omega) \subset \widehat{X}(\omega)\}) = 1$. This proves (i). We know $\{F_n\}$ is also a stationary process from Corollary 4.1. The proof of (ii) follows from identical arguments made in the proof of (i). ∎

Proof of Theorem 4.1. Since g is uniformly continuous, and $\phi_\omega(n+1, n, \cdot) = g(\xi_n(\omega), \cdot)$ it follows that the process ϕ_ω has equicontinuity for all $\omega \in \Omega$. Further by Lemma 4.4, $X_n(\omega) \subset \widehat{X}(\omega)$ for all n and $F_n(\omega) \subset \widehat{F}(\omega)$ for all n with probability 1. Lemmas 3.3 and 4.4 together imply statement (i) of Theorem 4.1 holds for all ω belonging to a set of probability 1. From Lemma 4.4 and (21), we can also find subset of Ω of probability 1 such that the process ϕ_ω satisfies the hypotheses (a), (b), and (c) of Lemma 3.1. The statements (ii), (iii), and (iv) of Theorem 4.1 follow from applying Lemma 3.1 for each such process ϕ_ω. Statement (v) follows from Proposition 2.3. ∎

5. Deterministic base relation: Ergodic input. We consider (1) with $\{\xi_n\}$ as an ergodic process in this section, and prove in Theorem 5.1 the result that is essentially in (v) of Theorem 1.2. The reader may note that it is just not enough to show that the base relation is alone identical with probability 1 in Theorem 5.1, since we had assumed that subspace topology on the base domain.

Theorem 5.1. *Let U be a complete metric space and $\xi_n : \Omega \to U$ a stationary process defined on a probability space $(\Omega, \mathcal{F}, P)$. Also, let $g : U \times X$ be any continuous map, where X is a compact metric space and let g define an RDE as in* (1). *Consider the set of all NDEs on X obtained by $g_{n,\omega}(\cdot) := g(\xi_n(\omega), \cdot)$, where $\omega \in \Omega$. Now, if*

(a) *$\{X_n(\omega)\}$ is the natural association of $\{g_{n,\omega}(\cdot)\}$, and*

(b) *ξ_n is an ergodic process,*

then there exist a set $\widetilde{X} \subset X$ and a relation $\widetilde{F} \subset \widetilde{X} \times \widetilde{X}$ such that for all ω belonging to a set of probability 1*,*

(i) *the base domain $\widehat{X}(\omega) = \widetilde{X}$, and*

(ii) *the base relation $\widehat{F}(\omega) = \widetilde{F}$.*

Lemma 5.1 provides an alternative representation of the base domain and a base repeller to that made in Definition 2.7.

Lemma 5.1. *Let ϕ be a process on a compact space X, $\{X_n\}$ its natural association, $\widehat{X}$ its base domain, and $\widehat{F}$ its base relation. Define the function $Y_\phi : \mathbb{Z} \to 2^X$ by*

$$Y_\phi(k) := \limsup_{j \geq k,\ j \to \infty} \phi(j, k, X)$$

and $h_\phi : \mathbb{Z} \to 2^{X \times X}$ by

$$h_\phi(k) := \limsup_{j \geq k,\ j \to \infty} \{(x, \phi(j+1, j, x)) : x \in X_{j,k}\},$$

where $X_{j,k} = \phi(j, k, X)$. Then

(a) *$Y_\phi(k) \in \mathsf{H}_X$ and $Y_\phi(k) \subset Y_\phi(k+1)$ for all k, and*

$$\widehat{X} = \bigcap_n Y_\phi(n); \tag{27}$$

(b) *$h_\phi(k) \in \mathsf{H}_{X \times X}$ and $h_\phi(k) \subset h_\phi(k+1)$ for all k, and*

$$\widehat{F} = \bigcap_n h_\phi(n). \tag{28}$$

Proof. (i) Fix any k. Clearly, from the definition of ϕ, the set $\phi(j, k, X)$ is nonempty for all $j \geq k$. Since X is compact $Y_\phi(k)$ is nonempty for all k. By definition of lim sup of sets, $Y_\phi(k)$ is closed in X. Also, $\phi(j,k{+}1,X) \supset \phi(j,k,X)$ for all $j \geq k{+}1$. By taking lim sup, $Y_\phi(k) \subset Y_\phi(k+1)$.

We now prove (27). We show $\subset$ in (27). Since $\{X_n\}$ is an association, $X_j = \phi(j,k,X_k) \subset \phi(j,k,X)$ for any $k \leq j$. Hence

$$\limsup_{j\geq k,\ j\to\infty} X_j \subset \limsup_{j\geq k,\ j\to\infty} \phi(j,k,X).$$

Thus $\widehat{X} \subset Y_\phi(k)$ which proves $\subset$ in (27).

We now show $\supset$ in (27). It follows directly from the definition of lim sup that for any two sequences of sets $\{E_{j,1}\}$ and $\{E_{j,2}\}$, $\limsup_{j\to\infty} E_{j,1} \cap \limsup_{j\to\infty} E_{j,2} \supset \limsup_{j\to\infty}(E_{j,1} \cap E_{j,2})$. This also holds for countable intersections, i.e.,

$$\limsup_{j\to\infty} E_{j,1} \cap \limsup_{j\to\infty} E_{j,2} \cap \cdots \supset \limsup_{j\to\infty}(E_{j,1} \cap E_{j,2} \cap \cdots). \tag{29}$$

Applying this, we have

$$\begin{aligned}\bigcap_k Y_\phi(k) = \bigcap_k \limsup_{j\geq k,\ j\to\infty} \phi(j,k,X) &\overset{(29)}{\supset} \limsup_{j\to\infty} \bigcap_{k\leq j} \phi(j,k,X) \\ &= \limsup_{j\to\infty} X_j.\end{aligned}$$

(ii) The idea is the same as in the proof of (i). We provide the essential steps. The fact that $X_{j,k}$ is nonempty for all $j \geq k$ and that since $X \times X$ is compact, the lim sup in the definition of $h_\phi(k)$ ensures $h_\phi(k)$ is nonempty.

Since $X_{j,k} \subset X_{j,k+1}$ for all $j \geq k+1$, we have $\{(x,\phi(j+1,j,x)) : x \in X_{j,k}\} \subset \{(x,\phi(j+1,j,x)) : x \in X_{j,k+1}\}$ for all $j \geq k+1$. Taking lim sup gives $h_\phi(k) \subset h_\phi(k+1)$. We now show $\subset$ in (28). Recall that $\widehat{F} = \limsup_{j\to\infty}\{(x,\phi_\omega(j+1,j,x)) : x \in X_j\}$. Since $X_j \subset X_{j,k}$ for any $k \leq j$, we have $\widehat{F} \subset h_\phi(k)$ for all k. Hence $\subset$ in (28) holds.

We now show $\supset$ in (28). Recalling the definition of $\widehat{h}_k(\omega)$,

$$\begin{aligned}\bigcap_k h_\phi(k) &= \bigcap_k \limsup_{j\geq k,\ j\to\infty} \{(x,\phi(j+1,j,x)) : x \in X_{j,k}\} \\ &\overset{(29)}{\supset} \limsup_{j\to\infty} \bigcap_{k\leq j} \{(x,\phi(j+1,j,x)) : x \in X_{j,k}\} \\ &= \limsup_{j\to\infty} \{(x,\phi(j+1,j,x)) : x \in X_j\} \\ &= \widehat{F}. \qquad \square\end{aligned}$$

LEMMA 5.2. *Let* $Y_{\phi_\omega}(k) := \limsup_{j\geq k,\ j\to\infty} \phi_\omega(j,k,X)$ *and* $h_{\phi_\omega}(k) := \limsup_{j\geq k,\ j\to\infty}\{(x,\phi_\omega(j+1,j,x)) : x \in X_{j,k}(\omega)\}$, *where* $X_{j,k}(\omega) = \phi_\omega(j,k,X)$. *Then* $Y_\phi(k) : \Omega \to \mathsf{H}_X$ *and* $h_\phi(k) : \Omega \to \mathsf{H}_{X\times X}$ *is measurable for each* k.

Proof. The proof is similar to that of Lemma 4.2. $\square$

LEMMA 5.3. *If* ξ_n *in Theorem* 5.1 *is ergodic, then* $\{Y_{\phi_\omega}(k)\}$ *and* $\{h_{\phi_\omega}(k)\}$ *are* H_X *and* $\mathsf{H}_{X\times X}$ *valued ergodic processes.*

Proof. The proof is similar to that of Corollary 4.1. $\square$

LEMMA 5.4 (see [11]). *If* $T : X \to X$ *is a measure-preserving transformation of the probability space* $(X,\mathcal{B},m)$, *then the following statements are equivalent:*

(i) T *is ergodic;*
(ii) *for every* $A \in \mathcal{B}$ *with* $m(A) > 0$ *we have* $m(\bigcup_{n=1} T^{-n}(A)) = 1$.

LEMMA 5.5. *Consider the random variables $Y_\phi(k)$ and $h_\phi(k)$ defined in Lemma 5.2. Then there exist closed Borel subsets $\widetilde{X} \subset X$ and $\widetilde{F} \subset X \times X$, such that $Y_{\phi_\omega}(k) = \widetilde{X}$ and $h_{\phi_\omega}(k) = \widetilde{F}$ for all ω belonging to a set of probability 1 and for all k.*

Proof. From Lemma 5.3, $Y = \{Y_\phi(k)\}$ is an ergodic process and hence a stationary process. Let the corresponding MPDS be $(\mathsf{H}_X^\infty, \mathcal{B}^\infty, P_Y, \sigma)$, with its different constituents taking the same fixed semantics in the standard way of obtaining an MPDS from a stationary process.

We now claim that there exists a Borel subset $\widetilde{X}$ of H_X such that $Y_{\phi_\omega}(k) = \widetilde{X}$ for all ω belonging to a set of probability 1 and for all k. Assume this were not true, i.e., assume that there exists no Borel subset $\widetilde{X}$ of H_X such that $P(\{\omega : Y_{\phi_\omega}(k) = \widetilde{X} \forall k\}) = 1$. Then we can find a Borel subset A of H_X such that $0 < P(\{\omega : Y_{\phi_\omega}(k) \subset A \forall k\}) < 1$ and $0 < P(\{\omega : Y_{\phi_\omega}(k) \subset A^c \, \forall k\}) < 1$. Consider the cylinder set $B \in \mathcal{B}^\infty$ defined by:

$$B := (\cdots \times \mathsf{H}_X \times A \times \mathsf{H}_X \times \cdots)$$

with A in the k_0th position. By definition $P_Y(\cdot) = P(Y^{-1}(\cdot))$. Hence $P_Y(B) = P(\{\omega : Y_{\phi_\omega}(k_0) \subset A\})$ and thus $0 < P_Y(B) < 1$.

Let $\bar{p} \in \mathsf{H}_X^\infty$ be denoted by $\bar{p} = (\cdot, p_{-1}, p_0, p_1, \ldots)$. Define the subset $E \subset \mathsf{H}_X^\infty$ such that $E := \{\bar{p} : \exists \text{ no } \omega \text{ s.t. } Y_{\phi_\omega}(k) = p_k \, \forall k\}$. By the definition of P_Y in the MPDS, $P_Y(E) = 0$.

Denote $B' := B \cap E^c$. Now, since $0 < P_Y(B) < 1$ and $P_Y(E) = 0$, we have $0 < P_Y(B') < 1$. By the definition of B', for each $\bar{p} \in B'$, there exists $\omega \in \Omega$ such that $p_k = Y_{\phi_\omega}(k)$ for all k. But $Y_{\phi_\omega}(k) \subset Y_{\phi_\omega}(k+1)$ from Lemma 5.1 for any ω and any k. Hence $p_k \subset p_{k+1}$ for all k. This implies that $\sigma(\bar{p}) \supset \bar{p}$, where σ is the left shift map on H_X^∞. In other words, $\sigma^{-1}(\bar{u}) \subset \bar{u}$. Thus $\sigma^{-1}(B') \subset B'$. Since $\sigma^{-1}(B') \subset B'$, we have $\bigcup_{n=1}^\infty \sigma^{-n}(B') \subset B'$. Applying Lemma 5.4 we have $P_Y(\bigcup_{n=1}^\infty \sigma^{-n}(B')) = 1$. Since $\bigcup_{n=1}^\infty \sigma^{-n}(B') \subset B'$, $P_Y(B') = 1$. This is a contradiction. Hence there exists a Borel subset $\widetilde{X}$ of H_X such that $P(\{\omega : Y_{\phi_\omega}(k) = \widetilde{X} \, \forall k\}) = 1$.

The proof that there exists a Borel subset $\widetilde{F}$ of $\mathsf{H}_{X \times X}$ such that

$$P(\{\omega : h_{\phi_\omega}(k) = \widetilde{F} \, \forall k\}) = 1$$

is similar. ◻

Proof of Theorem 5.1. Let $Y_{\phi_\omega}(k) = \limsup_{j \geq k,\, j \to \infty} \phi_\omega(j, k, X)$ and $h_{\phi_\omega}(k) = \limsup_{j \geq k,\, j \to \infty} \{(x, \phi_\omega(j+1, j, x)) : x \in X_{j,k}(\omega)\}$, where $X_{j,k}(\omega) = \phi_\omega(j, k, X)$. Applying Lemma 5.1 to each ω we have

$$\widehat{X}(\omega) = \bigcap_k Y_{\phi_\omega}(k) \text{ and } \widehat{F}(\omega) = \bigcap_k h_{\phi_\omega}(k). \tag{30}$$

By Lemma 5.5, we obtain $\bigcap_k Y_{\phi_\omega}(k) = \widetilde{X}$ and $\bigcap_k h_{\phi_\omega}(k) = \widetilde{F}$, where $\widetilde{X} \in \mathsf{H}_X$ and $\widetilde{F} \in \mathsf{H}_{X \times X}$ for all ω with probability 1. Using this in (30), we obtain (i) and (ii) of Theorem 5.1. ◻

6. Discussion. The highlight of this paper is in the identification of autonomous subsets of the phase space in which were found various nonautonomous attractors of an NDE generated by a typical ω of the RDE in (1). This was done via remodeling the NDE via a closed relation. In particular, when $\{\xi_n\}$ is ergodic, we obtain the *typical base relation* as a remodel of the entire RDE since all typical realizations of $\{\xi_n\}$ yield an identical base relation (Theorem 5.1). In that case all the interesting dynamics of the RDE would be contained in the chain-recurrent set of the typical base relation

Table 1
A comparison of attractor concepts.

	Crauel, Duc, and Siegmund [7]	Liu [13]	Our paper
Scope	random flows	semiflows in continuous time	continuous random flows and semiflows in discrete time with stationary noise (input)
Convergence	in probability	almost surely	almost surely
Pullback attractor's nature	invariant	+invariant	both types, +invariant and invariant
Uniformity of attractors	not known	not known	+invariant attractor is uniform

(due to (iv) of Theorem 1.2). Thus, effectively, there is a *deterministic autonomous subset* which captures all the interesting dynamics of an RDE with an ergodic input. Based on nonautonomous attractors, we undertake a conceptual comparison of the attractors proven here with that available in the literature. The attractors defined in our work add to the already available list of nonequivalent definitions [18] of random attractors. Table 1 offers a quick comparison of only some core properties of known local attractor concepts, but it ignores comparison between other fine properties. To conclude, we hint on the potential usage of our closed relations approach. We restrict ourselves to the case where the stochastic input in the RDE is from an ergodic source and use the typical base relation for explanation. With an elementary application of an *ergodic theorem like the Birkhoff's ergodic theorem*, one can easily show that $\{(\vartheta_i, \vartheta_{i+1}) : i \in \mathbb{N}$ and $\{\vartheta_n\}$ is an entire solution of $\{g_{n,\omega}\}\}$, the subset of $X \times X$ is in fact the typical base relation for almost all realizations of ω. Thus, whenever an experimental time series is the only information of an RDE model available, it is possible to get, from an ensemble of a finite length time series, an estimate of the typical base relation. Once the typical base relation is estimated, simple heuristic algorithms on a computer can even be designed to find rough estimates of attractor blocks (owing to their simple definition) of a relation provided the dimension of X is within the reach of computing power. Keeping in mind that there exists an attractor in every attractor block and every neighborhood of any attractor of a closed relation contains an attractor block, a rough estimate of the attractor itself can be obtained by attempting to find smaller attractor blocks. Since, from the typical base-relation estimate, the estimate of the transpose of the base relation is easily obtained, the base repellers of the typical base relation can also be estimated. These estimates of attractors and repellers lead to a comprehensive picture of the dynamics owing to the Conley decomposition theorem.

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