Laboratory Games and Quantum Behaviour: The Normal Form with a Separable State Space

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Abstract

To describe quantum behaviour, Kolmogorov's definition of probability is extended to accommodate subjective beliefs in a particular "laboratory game" that a Bayesian rational decision maker plays with Nature, Chance and an Experimenter. The Experimenter chooses an orthonormal subset of a complex Hilbert space of quantum states; Nature chooses a state in this set along with an observation in a measurable space of experimental outcomes that influences Chance's choice of random consequence. Imposing quantum equivalence allows the trace of the product of a density and a likelihood operator to represent the usual Bayesian expectation of a likelihood function w.r.t. a subjective prior.

1 Introduction

1.1 Subjective Probability Measures

The first axiomatic foundation for expected utility appeared in an appendix to von Neumann and Morgenstern's *Theory of Games and Economic Behavior* (1953). Their focus was on "objective" or hypothetical probabilities, with definite numerical values. Savage (1954) extended the earlier ideas of Keynes, Ramsey and de Finetti in order to provide an axiomatic justification for maximizing *subjective* expected utility (SEU). Later, Anscombe and Aumann (1963) devised a more transparent approach that combines: (i) "roulette lotteries" having objective probabilities; (ii) "horse lotteries" that, as their theory implies, should be given subjective probabilities.[1] Their ideas have been elaborated by Fishburn (1982) and Hammond (1998b) in order to derive a countably additive subjective probability measure over a single general measurable space of uncertain events. The latter builds on Hammond (1998a) in order to derive many of the standard rationality axioms in Bayesian decision theory using a "consequentialist" perspective intended to embrace von Neumann and Morgenstern's claim that no generality is lost in reducing an extensive form game to its normal form.[2] This paper, accordingly, confines its attention to analysing the SEU hypothesis in a "laboratory" game in normal form that is rich enough to describe quantum behaviour observed in many experiments.

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[1]Following earlier work on qualitative probability, a crucial axiom in Savage's theory explicitly requires that preferences be consistent with events being ordered by their likelihood. Lehrer and Shmaya (2006) use a similar axiom in their approach to subjective quantum probability. Hammond (1998b) discusses why it may be desirable to follow Anscombe and Aumann in not imposing such an axiom directly.

[2]Actually, much of modern game theory rejects this claim, with two major exceptions: (i) any two-person zero-sum game; (ii) any single-person decision problem of the kind that our laboratory games reduce to.

1.2 Quantum Bayesian Decision Theory

Heisenberg's uncertainty principle is an often cited example where setting up an experimental apparatus intended to observe one variable, such as a particle's position, may make it impossible to observe simultaneously some other variable, such as the same particle's momentum. Physical laws force a choice of what to observe — either position, or momentum, or perhaps some combination of the two. Similarly, in the two-slit laboratory experiment famously discussed by Feynman (1951), if the apparatus can detect which slit the particle passes through, that destroys whatever wave-like interference effects would otherwise have been observed.

Even so, several prominent "quantum Bayesians" have set out to identify quantum states with the subjective probabilities that feature in Bayesian decision theory.[3] Their task has remained far from straightforward, however, because quantum behaviour can only be fully described within an extended probability system that allows the experiment to influence the measurable space of events on which probabilities are defined. Up to now, indeed, quantum Bayesians typically consider only: (i) a finite-dimensional Hilbert space of possible quantum states, which obviously excludes many important physical phenomena; (ii) a special model involving what economists call a "risk-neutral" agent — i.e., one whose decisions maximize subjective expected wealth rather than the subjective expected utility (SEU) of wealth that is typical in modern Bayesian decision theory, and that allows the agent's preferences to be risk averse or even risk seeking.

This paper begins an attempt to extend both general measure-theoretic Bayesian decision theory and Bayesian inference so that they can accommodate uncertainty caused by quantum phonomena. For now we limit our analysis to a separable Hilbert space where operators have matrix representations;[4] non-separable Hilbert spaces allowing general "likelihood operators", whose spectral decompositions may require integrals rather than countable sums, are left for later work.[5]

1.3 Outline of Paper

Section 2 sets out a four-person "laboratory game".[6] Its first three players are the Decision Maker ($\mathbf{D}$), Nature ($\mathbf{N}$), and Chance ($\mathbf{C}$), who have essential roles in any non-trivial game that is consistent with the Anscombe–Aumann approach to subjective probability. But here they are joined by a fourth player: an Experimenter ($\mathbf{E}$) whose strategy, possibly in the form of a choice of physical apparatus, determines (perhaps inadvertently) both: a) a measurable subspace representing what kinds of function can be observed; b) what probability law governs the observable result of the experiment. In this setting, SEU theory requires player $\mathbf{D}$ to have a subjective probability measure defined on an appropriate σ-algebra over the space of triples (e, s, x) that combine: (i) player $\mathbf{E}$'s choice of experiment $e \in E$; (ii) player $\mathbf{N}$'s choice of state s consistent with e; (iii) the observed result of the experiment x that then informs player $\mathbf{C}$'s choice of random consequence.

Indeed, different choices by $\mathbf{E}$ will typically induce not only different σ-algebras, as in Vorob′ev (1962), but also different measurable subspaces of Nature's strategy space.[7] Even so, although in

[3] See in particular Schack, Brun and Caves (2001), Caves, Fuchs and Schack (2002), and Fuchs and Schack (2009), as well as the extensive work they cite.

[4] Recall that a topological space is separable if there is a countable subset whose closure is the whole space.

[5] Danilov and Lambert-Mogiliansky (2008, 2010) consider partially ordered sets in a more general framework that admits as a special case orthonormal subsets of (finite-dimensional) Hilbert space. La Mura (2009) limits himself to "projective" expected utility on finite-dimensional Euclidean space.

[6] There are key differences from the games considered by Shafer and Vovk (2001, pp. 189–191) or Pitowsky (2003).

[7] Hess and Philipp (2005) appear to have been first to realize how relevant Vorob′ev's insufficiently appreciated work is to quantum theory, though Pitowsky (1994) does note Boole's (1862) related ideas from exactly 100 years earlier. See also Khrennikov (2008). Somewhat similar ideas appear in work by Slavnov (2001) and by Janssens (2004), who nevertheless seem uanware of Vorob′ev's contribution.

principle player $\mathbf{D}$'s subjective probability of any event could depend on which experiment $e \in E$ is used to detect its presence or absence, we follow Vorob′ev (1962) in postulating that it does not.

Following this discussion of general experiments, section 3 specifies the laboratory game in a quantum setting. Each state chosen by Nature is identified with a normalized *hidden stochastic parameter* vector $\mathfrak{h}$ in the unit sphere $\mathbb{S}$ of a separable Hilbert space $\mathcal{H}$ over the complex field $\mathbb{C}$. Then $\mathbf{E}$'s choice of experiment is assumed to determine an orthonormal subset of $\mathcal{H}$. Quantum theory, however, treats as one event all orthonormal subsets of $\mathcal{H}$ that are equivalent in the sense of having the same closed linear span. When Gleason's (1957) theorem applies — as it does provided $\mathcal{H}$ has dimension at least 3 — there must be both density and likelihood operators on $\mathcal{H}$ that together allow probabilities in particular and expectations in general to be calculated, whenever they are defined, using the familiar trace rule of quantum mechanics.

The brief final Section 4 contains concluding remarks and an agenda for further research.

2 A Four-Person Laboratory Game

2.1 Players, Strategies, and Payoffs

In effect, the decision problems which Anscombe and Aumann (1963) consider involve a general consequence domain Y, a finite set A of possible actions a, and a finite set S of possible states of nature s. Our laboratory game is considerably more general, with measurable spaces $(X, \mathcal{B})$ and $(S, \mathcal{S})$ whose elements are, respectively, directly observable experimental outcomes x, and states of nature s that are unobservable latent parameters of the relevant probability law that determines x. We assume that states can at best be inferred only indirectly and incompletely from the observed value of x.

To avoid inessential measurability issues, we restrict attention to a finite consequence domain Y. To avoid trivialities, we assume that Y has at least two elements, one of which is superior to the other. We also assume that the space $(X, \mathcal{B})$ is *Polish* — i.e., metrizable by some metric $d : X \times X \to \mathbb{R}_+$ that makes (X, d) a complete separable metric space, with $\mathcal{B}$ as its Borel σ-algebra. Let $\Delta(X, \mathcal{B})$ denote the set of Kolmogorov probability measures on $(X, \mathcal{B})$.

As mentioned in the introduction, we consider a game with the following four players:

D is a Decision Maker, who chooses an optimal decision strategy or action $a \in A$.

E is an Experimenter, whose choice of experiment $e \in E$ determines a measurable space $(S_e, \mathcal{S}_e)$ of (hidden) states, where $S_e \in \mathcal{S}$ and $\mathcal{S}_e \subseteq \mathcal{S} \cap \{S_e\} := \{S' \cap S_e \mid S' \in \mathcal{S}\}$;

N is Nature, who chooses any mapping $E \ni e \mapsto (s_e, x_e) \in S_e \times X$ that, for each $e \in E$, determines both the state s_e and the experimental outcome x_e in a "horse lottery".

C is Chance who, after being informed of both player $\mathbf{D}$'s action $a \in A$ and the experimental observation $x \in X$, but not knowing anything more about either the state $s \in S$ or the experiment $e \in E$, sets up a "roulette lottery" with random consequences $y \in Y$, whose probability distribution $y \mapsto \lambda(a, x; y)$ player $\mathbf{D}$ knows, and where, for each fixed $a \in A$ and $y \in Y$, the map $x \mapsto \lambda(a, x; y)$ is $\mathcal{B}$-measurable.

To complete the description of the game, we must specify the four players' payoff functions. In fact players $\mathbf{E}$, $\mathbf{N}$ and $\mathbf{C}$ will be treated as passive, meaning that their payoffs are arbitrary constants, entirely independent of the game's outcome. Player $\mathbf{D}$, however, is assumed to have an expected payoff which depends on the action a and experimental outcome x through the probability law $(a, x, y) \mapsto \lambda(a, x; y)$, but does not depend directly on either the experiment e or the state s.

2.2 Subjective Expected Utility (SEU)

In the context of this laboratory game, applying an SEU axiom system such as that of Fishburn (1982) or (very closely related) Hammond (1998) implies that player $\mathbf{D}$ must have:

1. a unique cardinal equivalence class of von Neumann–Morgenstern utility functions (NMUFs) $Y \ni y \mapsto u(y) \in \mathbb{R}$;[8]

2. a unique Bayesian prior probability measure $\mathcal{F} \ni K \mapsto \pi(K) \in [0,1]$, where $S := \cup_{e \in E} S_e$ and $\mathcal{F}$ is the σ-algebra of all sets $F \subseteq E \times S \times X$ whose sections

$$F_e := \{(s,x) \in S \times X \mid (e,s,x) \in F\} \tag{1}$$

are $\mathcal{S}_e \otimes \mathcal{B}$-measurable subsets of $S_e \times X$, for all $e \in E$;[9]

3. a unique preference ordering[10] over A represented by the well-defined SEU integral function

$$A \ni a \mapsto U(a) := \int_{E \times S \times X} \left[\sum\nolimits_{y \in Y} \lambda(a,x;y)\, u(y)\right] d\pi. \tag{2}$$

Define $p_e := \pi(\{e\} \times S_e \times X)$ as the probability that player $\mathbf{E}$ chooses experiment $e \in E$. Then, given any $e \in E$ with $p_e > 0$, define $Q_e(J) := \pi(\{e\} \times J)/p_e$ so that $\mathcal{S}_e \otimes \mathcal{B} \ni J \mapsto Q_e(J)$ is the conditional probability measure on $S \times X$. Because $(X, \mathcal{B})$ is Polish, the measure $\mathcal{F} \ni F \mapsto \pi(F) \in [0,1]$ has values given by the composition

$$\pi(F) = \sum\nolimits_{e \in E} p_e\, Q_e(F_e) = \sum\nolimits_{e \in E} p_e \int_{S_e} \left[\int_X 1_F(e,s,x)\, \xi_e(dx|s)\right] P_e(ds) \tag{3}$$

where, given any $e \in E$ with $p_e > 0$, Nature chooses both:

1. a random state $s \in S_e$ according to a **Bayesian prior** probability measure $P_e(ds)$ on $(S_e, \mathcal{S}_e)$ satisfying $P_e(G) = \pi(\{e\} \times G \times X)/p_e$ for each $G \in \mathcal{S}_e$.

2. an observed experimental result x according to the **Bayesian likelihood** law in the form of a *regular conditional distribution* (rcd) $S_e \ni s \mapsto \xi_e(dx|s) \in \Delta(X, \mathcal{B})$ which is $\mathcal{S}_e$-measurable and satisfies

$$\int_{S_e} \left[\int_X 1_F(e,s,x)\, \xi_e(dx|s)\right] P_e(ds) = Q_e(F_e) = \frac{\pi(\{e\} \times F_e)}{p_e} \quad \text{for all } F \in \mathcal{F}. \tag{4}$$

For each experiment $e \in E$, the marginal of Q_e on X is a probability measure on $(X, \mathcal{B})$ that we denote by ξ_e^*. For any event $B \in \mathcal{B}$, it equals the expected value

$$\mathcal{B} \ni B \mapsto \xi_e^*(B) := Q_e(S_e \times B) = \int_{S_e} \xi_e(B|s)\, P_e(ds), \tag{5}$$

of the likelihood $\xi_e(B|s)$ w.r.t. the prior $P_e(ds)$ on $(S_e, \mathcal{S}_e)$.

[8]Two NMUFs $u, \tilde{u} : Y \to \mathbb{R}$ are *cardinally equivalent* iff there exist both an additive constant $\alpha \in \mathbb{R}$ and a positive multiplicative constant $\gamma \in \mathbb{R}$ such that $\tilde{u}(y) \equiv \alpha + \gamma u(y)$ on Y.

[9]We emphasize that π is a countably additive probability *measure*, as in Arrow's (1970) and Fishburn's (1982) refinements of Savage's (1954) theory. By contrast, Savage derived only finitely additive probability, as do Gyntelberg and Hansen (2009) in the quantum context.

[10]A *preference ordering* on A is a binary relation $\succsim$ on A that is complete and transitive. It is *represented* by $U : A \to \mathbb{R}$ just in case $U(a) \geq U(a')$ iff $a \succsim a'$, for all $a, a' \in A$.

In turn, each of player $\mathbf{D}$'s possible actions $a \in A$ generates the unconditional roulette lottery

$$Y \ni y \mapsto \lambda_e^*(a; y) := \int_X \lambda(a, x; y)\, \xi_e^*(dx) = \int_{S_e} \left[\int_X \lambda(a, x; y) \xi_e(dx|s) \right] P_e(ds) \tag{6}$$

on Y whose expected utility to player $\mathbf{D}$, given the experiment $e \in E$, is determined by the SEU function

$$A \ni a \mapsto V_e(a) := \sum\nolimits_{y \in Y} \lambda_e^*(a; y)\, u(y). \tag{7}$$

This decomposition allows the SEU formula (2) to be rewritten as $U(a) = \sum_{e \in E} p_e V_e(a)$.

2.3 Vorob′ev Consistent Extended Probability Systems

The subjective probability structure set up in Section 2.2 places no restrictions at all on either the subjective prior probabilities P_e of states $s \in S$ in the measurable spaces $(S_e, \mathcal{S}_e)$ corresponding to different experiments $e \in E$, or on the associated likelihood laws $s \mapsto \xi_e(dx|s) \in \Delta(X, \mathcal{B})$. Yet presumably good experiments should do as little as possible to perturb either of these. Extending an idea due to Vorob′ev (1962) so that we can meet the needs of quantum mechanics, the only change in both the probability measures $\mathcal{S}_e \ni G \mapsto P_e(G)$ and the $\mathcal{S}_e$-measurable likelihood law $S_e \ni s \mapsto \xi_e(dx|s)$ for different experiments $e \in E$ will be in their domain of definition $(S_e, \mathcal{S}_e)$.[11] Specifically, there must exist a set function $P : \mathcal{S}^* \to [0, 1]$ and likelihood law $S \ni s \mapsto \xi(dx|s)$, both independent of e, such that: (i) $P(G) = P_e(G)$ whenever $G \in \mathcal{S}_e$; (ii) $\xi(B|s) = \xi_e(B|s)$ whenever $s \in \mathcal{S}_e$; (iii) for each $B \in \mathcal{B}$ and $e \in E$, the restriction of $s \mapsto \xi_e(dx|s)$ to S_e must be $\mathcal{S}_e$-measurable. In particular, we have a **Vorob′ev consistent extended probability system**

$$(S, \{(S_e, \mathcal{S}_e)\}_{e \in E}, P) \tag{8}$$

where each $(S_e, \mathcal{S}_e)$ is a measurable space, and $S = \cup_{e \in E} S_e$.

It should be noted that, given any pair G, G' of disjoint subsets of S, the usual additivity condition $P(G \cup G') = P(G) + P(G')$ needs to hold *only if* there is at least one experiment $e \in E$ such that both G and G' belong to the *same* space $(S_e, \mathcal{S}_e)$.[12] Furthermore, given any countable[13] collection $\{G_n\}_{n \in N}$ of pairwise disjoint subsets of S, the usual countable additivity condition

$$P\left(\bigcup\nolimits_{n \in N} G_n \right) = \sum\nolimits_{n \in N} P(G_n) \tag{9}$$

must hold *if* there exists at least one experiment $e \in E$ such that $G_n \in \mathcal{S}_e$ for all $n \in N$.

Indeed, it is instructive to compare Vorob′ev consistency with a stronger condition for the existence of a Kolmogorov extension. The latter would require $P(G)$ to be defined for all G in $\sigma(\mathcal{S}^*)$, the smallest σ-algebra containing all the sets in $\mathcal{S}^*$, which is typically much larger than $\mathcal{S}^*$; moreover, (9) must hold for any collection $\{G_n\}_{n \in N}$ of pairwise disjoint subsets in $\sigma(\mathcal{S}^*)$.

Finally, for future reference we record how Vorob′ev consistency implies that, for each $e \in E$, the probability measure $\mathcal{S}_e \otimes \mathcal{B} \ni J \mapsto Q_e(J)$ on $S_e \times X$ defined by (4) must satisfy

$$Q_e(J) = \int_{S_e} \left[\int_X 1_J(s, x)\, \xi(dx|s) \right] P(ds). \tag{10}$$

Of course, the probability measure $J \mapsto Q_e(J)$ depends on e only through the domain $\mathcal{S}_e \otimes \mathcal{B}$ of measurable sets J, precisely as Vorob′ev consistency requires.

[11] Vorob′ev (1962) considered a special case where the family of measures takes the form $(S, \{\mathcal{S}_e\}_{e \in E}, P)$ with a varying σ-algebra $\mathcal{S}_e$ which, however, always includes the entire space S as its largest member.

[12] This is an obvious extension to σ-algebras of Griffiths' (2003) "single-framework rule" for Boolean algebras.

[13] We regard any finite set as countable, as well as any countably infinite set.

3 Application to Quantum Experiments

3.1 Extended Probability over Quantum States

In the quantum version of our four-person laboratory game, players $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{C}$ have the same strategy spaces. Moreover, player $\mathbf{C}$'s chosen probability law $(a, x, y) \mapsto \lambda(a, x; y)$ still describes the random consequence of the game. Player $\mathbf{N}$, however, is now assumed to have a specific strategy set S equal to the unit sphere $\mathcal{S}$ of a particular physically relevant separable Hilbert space $\mathcal{H}$ over the complex field $\mathbb{C}$. This formulation, of course, is chosen so that solutions to Schrödinger's wave equation can be represented more easily.

Recall that a set $G \subset \mathcal{H}$ is *orthonormal* iff the inner products of all its pairs $\mathfrak{h}, \mathfrak{h}' \in G$ satisfy $\langle \mathfrak{h}, \mathfrak{h}' \rangle = \delta_{\mathfrak{h}\mathfrak{h}'}$, where $G \times G \ni (\mathfrak{h}, \mathfrak{h}') \mapsto \delta_{\mathfrak{h}\mathfrak{h}'} \in \{0, 1\}$ is the Kronecker delta function. Moreover, any orthonormal set G is a linearly independent subset of $\mathcal{S}$. When $\mathcal{H}$ has finite dimension d, then $\#G \leq d$; in any case, separability of $\mathcal{H}$ implies that G is countable.[14]

Given any subset $G \subseteq \mathcal{H}$ (not necessarily orthonormal), let $\overline{\mathrm{sp}}\, G$ denote the closure in $\mathcal{H}$ of the linear subspace spanned by G; it is also a linear subspace of $\mathcal{H}$. An orthonormal set $G \subset \mathcal{H}$ is *complete*, or an *orthonormal basis* of $\mathcal{H}$, if $\overline{\mathrm{sp}}\, G = \mathcal{H}$ — i.e., for any $\mathfrak{h} \in \mathcal{H}$, there exist matching countable sets of basis elements $\{\mathfrak{e}_n\}_{n \in N} \subset G$ and of scalars $\{c_n\}_{n \in N} \subset \mathbb{C}$ such that: either (i) N is finite and $\mathfrak{h} = \sum_{n \in N} c_n \mathfrak{e}_n$; or (ii) $N = \mathbb{N}$ and the Hilbert norm $\|\mathfrak{h} - \sum_{n=1}^{k} c_n \mathfrak{e}_n\| \to 0$ as $k \to \infty$.

In our laboratory game, we suppose that the measurable subspace determined by each possible experiment $e \in E$ takes the form $(G_e, 2^{G_e})$ for some orthonormal set G_e, where 2^{G_e} denotes the power set of all subsets of G_e.[15] However, we consider what Vorob′ev consistency implies, not just for a finite collection of experiments, but for all measurable subspaces $(G, 2^G)$ as G varies over the entire (non-empty) set $\mathcal{G}$ of orthonormal subsets of $\mathcal{H}$. Then the domain of the probability map $G \mapsto P(G)$ must be all of $\mathcal{G}$. So our Vorob′ev consistent extended probability system (8) becomes

$$(\mathcal{S}, \{(G, 2^G)\}_{G \in \mathcal{G}}, P). \tag{11}$$

3.2 Quantum Equivalent Events

The usual physical formulation of quantum mechanics (QM) represents uncertain events by members of the set $\mathcal{L}$ of all closed linear subspaces of $\mathcal{H}$ or, equivalently, of the set $\mathcal{P}$ of all orthogonal projections of $\mathcal{H}$ onto such subspaces. In our framework, QM imposes extra conditions on (11) through a **quantum equivalence relation** $\sim$ defined on the set $\mathcal{G}$ of all orthonormal subsets of $\mathcal{H}$ by

$$G \sim G' \iff \overline{\mathrm{sp}}\, G = \overline{\mathrm{sp}}\, G'. \tag{12}$$

Thus, each orthonormal set $G \subset \mathcal{S}$ belongs to its own equivalence class $[G]$ of events, which corresponds to both the closed linear subspace $\overline{\mathrm{sp}}\, G \in \mathcal{L}$, and the associated orthogonal projector $\Pi_{[G]} \in \mathcal{P}$ mapping $\mathcal{H}$ onto this linear subspace. In fact there are obvious bijections

$$[G] \leftrightarrow \overline{\mathrm{sp}}[G] \leftrightarrow \Pi_{[G]} \tag{13}$$

between the three spaces: (i) $[\mathcal{S}] := \mathcal{S}/\!\sim$ of equivalence classes of orthonormal subsets of $\mathcal{H}$; (ii) $\mathcal{L}$ of closed linear subspaces of $\mathcal{H}$; and (iii) $\mathcal{P}$ of orthogonal projections onto closed subspaces in $\mathcal{L}$.

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[14] Friedman (1982, Lemma 6.4.7) proves concisely the equivalent property that any orthonormal basis is countable.

[15] Equivalently, each G_e is a set of mutually orthogonal one-dimensional projectors, as in Caves, Fuchs and Schack (2002) for the case when $\mathcal{H}$ is finite-dimensional.

3.3 Quantum Probability Systems

As G varies over the orthonormal sets in $\mathcal{G}$, the bijections (13) induce in turn bijections

$$2^G \leftrightarrow \mathcal{L}_G \leftrightarrow \mathcal{P}_G \tag{14}$$

between each: (i) σ-algebra 2^G; (ii) family $\mathcal{L}_G \subseteq \mathcal{L}$ of closed linear subspaces spanned by subsets of G; (iii) family $\mathcal{P}_G \subseteq \mathcal{P}$ of orthogonal projections onto the corresponding closed linear subspaces in $\mathcal{L}_G$. We take the liberty of describing both $\mathcal{L}_G$ and $\mathcal{P}_G$ as **quantum σ-algebras**.[16]

In addition to Vorob′ev consistency, an obvious additional requirement is the **quantum consistency** condition requiring that quantum equivalent events be given the same probability. Specifically, whenever $[G] = [G']$ because $\overline{\mathrm{sp}}\, G = \overline{\mathrm{sp}}\, G'$, we insist that $P(G) = P(G')$. Then the bijections (13) allow us to define set functions $\mathcal{L} \ni L \mapsto \nu(L) \in [0,1]$ and $\mathcal{P} \ni \Pi \mapsto \mu(\Pi) \in [0,1]$ so that $P(G) = \nu(\overline{\mathrm{sp}}\, G) = \Pi_{[G]}$ for all $G \in [G]$. These definitions imply the obvious bijections

$$(\mathcal{S}, \{(G, 2^G)\}_{G\in\mathcal{G}}, P) \leftrightarrow (\mathcal{L}, \{\mathcal{L}_G\}_{G\in\mathcal{G}}, \nu) \leftrightarrow (\mathcal{P}, \{\mathcal{P}_G\}_{G\in\mathcal{G}}, \mu) \tag{15}$$

between the extended probability system (11) and its images induced by the bijections (14).

Next, suppose that $\{\Pi_n\}_{n\in N}$ is any countable collection of pairwise orthogonal projectors — i.e., they satisfy $\Pi_i\Pi_j = 0$ whenever $i, j \in N$ with $i \neq j$. This is equivalent to their respective ranges $\{L_n\}_{n\in N}$ being pairwise orthogonal closed subspaces of $\mathcal{H}$. For each $n \in N$, let G_n be an orthonormal basis of the subspace L_n. Pairwise orthogonality of the subspaces L_n evidently implies that the union $G := \cup_{n\in N} G_n$ of these orthonormal bases is also an orthonormal set. Let $L := \overline{\mathrm{sp}}\, G$, and note that the orthogonal projection onto L satisfies $\Pi_{[G]} = \Pi_L = \sum_{n\in N} \Pi_n$. It follows that the mapping $\Pi \mapsto \mu(\Pi)$ satisfies the countable additivity condition

$$\mu\left(\sum\nolimits_{n\in N} \Pi_n\right) = \mu(\Pi_L) = P(G) = \sum\nolimits_{n\in N} P(G_n) = \sum\nolimits_{n\in N} \mu(\Pi_n) \tag{16}$$

whenever the projectors $\{\Pi_n\}_{n\in N}$ are pairwise orthogonal. Hence $\Pi \mapsto \mu(\Pi)$ meets Parthasarathy's (1992, p. 31) definition of a *probability distribution on* $\mathcal{P}$. A triple $(\mathcal{S}, \mathcal{P}, \mu)$, where $\mathcal{P} \ni \Pi \mapsto \mu(\Pi) \in [0,1]$ satisfies (16) for any countable collection $\{\Pi_n\}_{n\in N}$ of pairwise orthogonal projectors, will therefore be called a **quantum probability system**. It succinctly summarizes an extended probability system (11), as well as its equivalents given by (15), that happen to be both Vorob′ev and quantum consistent.

3.4 Quantum Likelihood Operators

Section 2.3 introduced the Vorob′ev consistent likelihood law $S \ni s \mapsto \xi(dx|s)$ that is $\mathcal{S}_e$-measurable on each set S_e, as well as the probability measure $J \mapsto Q_e(J)$ given by (10) on the product measurable space $(S_e \times X, \mathcal{S}_e \otimes \mathcal{B})$ of states and experimental observations. Because $\mathcal{H}$ is separable, each orthonormal set $G \subset \mathcal{S}$ must be countable, so these two have respective quantum counterparts:

1. the **quantum likelihood law** $\mathcal{S} \ni \mathfrak{h} \mapsto \xi(dx|\mathfrak{h})$ which specifies how the likelihood $\xi(B|\mathfrak{h})$ of any Borel set $B \subset X$ changes as $\mathfrak{h}$ varies over G;

2. the joint probability measure $2^G \otimes \mathcal{B} \ni J \mapsto q_G(J)$ on $G \times X$ given by

$$q_G(J) = \sum\nolimits_{\mathfrak{h}\in G} \left[\int_X 1_J(\mathfrak{h}, x)\, \xi(dx|\mathfrak{h})\right] P(\{\mathfrak{h}\}) = \sum\nolimits_{\mathfrak{h}\in G} \xi(J_{\mathfrak{h}}|\mathfrak{h}) P(\{\mathfrak{h}\}), \tag{17}$$

where $J_{\mathfrak{h}} := \{x \in X \mid (\mathfrak{h}, x) \in J\}$ is the appropriate section of the set J, for each $\mathfrak{h} \in G$.

[16] A well known result of quantum logic is that $\mathcal{L}$, or equivalently $\mathcal{P}$, can be given an orthomodular lattice structure. Each family $\mathcal{L}_G$, or equivalently $\mathcal{P}_G$, is then an orthocomplemented Boolean sublattice.

For each fixed orthonormal set $G \subset \mathcal{S}$, consider the set function

$$\mathcal{B} \ni B \mapsto \Xi_G(B) := \sum\nolimits_{\mathfrak{h}\in G} \xi(B|\mathfrak{h})\, \Pi_{[\{\mathfrak{h}\}]} \tag{18}$$

whose value is a positive linear combination of the family $\Pi_{[\{\mathfrak{h}\}]}$ ($\mathfrak{h} \in G$) of one-dimensional projectors on $\mathcal{H}$. For each $B \in \mathcal{B}$, this combination must be a self-adjoint or Hermitean (linear) operator $\mathcal{H} \ni \mathfrak{h} \mapsto \Xi_G(B)\,\mathfrak{h}$ that is *positive* — i.e., $\langle \mathfrak{h}, \Xi_G(B)\,\mathfrak{h}\rangle \geq 0$ for all $\mathfrak{h} \in \mathcal{H}$. We call $B \mapsto \Xi_G(B)$ a **quantum likelihood operator**. Because each $B \mapsto \xi(B|\mathfrak{h})$ is a probability measure on the Borel sets $B \subseteq X$, the countable additivity condition $\Xi_G(\cup_{m\in N} B_m) = \sum_{m\in N} \Xi_G(B_m)$ is satisfied whenever the countable family $\{B_m\}_{m\in N}$ is pairwise disjoint. Also $\Xi_G(\emptyset)$ and $\Xi_G(X)$ are equal to the null and identity operators on $\mathcal{H}$, respectively. For each fixed orthonormal set $G \subset \mathcal{S}$, this makes $B \mapsto \Xi_G(B)$ a **positive operator-valued measure** (or POVM) on the measurable space $(X, \mathcal{B})$.

Note that for each fixed Borel set $B \in \mathcal{B}$ and orthonormal set $G \subset \mathcal{S}$, the countable range $\Lambda_G(B) := \cup_{\mathfrak{h}\in G}\{\xi_{\mathfrak{h}}(B)\} \subset [0,1]$ of possible likelihood numbers must constitute the (pure point or discrete) spectrum of eigenvalues for the operator $\Xi_G(B)$. Indeed, each eigenvalue $\ell \in \Lambda_G(B)$ has its own eigenspace $L^\ell_G(B) := \overline{\mathrm{sp}}\,\Gamma^\ell_G(B)$, where $\Gamma^\ell_G(B) := \{\mathfrak{h} \in G \mid \xi_{\mathfrak{h}}(B) = \ell\}$. Of course eigenspaces associated with distinct eigenvalues must be orthogonal.

Finally, as in (18), we define a POVM $2^G \otimes \mathcal{B} \ni J \mapsto Q_G(J)$ for each orthonormal $G \subset \mathcal{S}$ by

$$Q_G(J) := \sum\nolimits_{\mathfrak{h}\in G} \xi(J_{\mathfrak{h}}|\mathfrak{h})\, \Pi_{[\mathfrak{h}]}. \tag{19}$$

This is the operator equivalent of the joint probability measure $J \mapsto q_G(J)$ on the product measurable space $(G \times X, 2^G \otimes \mathcal{B})$ given by (17).

3.5 Gleason's Theorem and the Trace Rule

Let $\{\mathfrak{e}_n\}_{n\in N}$ be any orthonormal basis of $\mathcal{H}$. Then the *trace* of any positive self-adjoint operator ρ is defined by $\mathrm{tr}\,\rho := \sum_{n\in N}\langle \mathfrak{e}_n, \rho\mathfrak{e}_n\rangle$ even if this sum of non-negative term diverges to $+\infty$; its value is preserved by applying the same unitary transformation to all the vectors in $\mathcal{H}$, which is equivalent to changing its orthonormal basis. A *density operator* on $\mathcal{H}$ is any positive operator satisfying $\mathrm{tr}\,\rho = 1$.

Suppose the separable space $\mathcal{H}$ has dimension $d \geq 3$. Then a corollary of Gleason's (1957) theorem due to Parthasarathy (1992, Theorem 9.18) establishes that, because (16) holds, there is a density operator ρ satisfying $\mu(\Pi) = \mathrm{tr}(\rho\,\Pi)$ for all projections $\Pi \in \mathcal{P}$. This implies the **trace rule**

$$P(G) = \mu(\Pi_{[G]}) = \mathrm{tr}(\rho\,\Pi_{[G]}) \text{ for every orthonormal set } G \subset \mathcal{S}. \tag{20}$$

Similar trace rules also apply to the joint probability measure $J \mapsto q_G(J)$ on $(G \times X, 2^G \otimes \mathcal{B})$, as well as to its marginal distribution $B \mapsto \xi^*_G(B) = Q_G(G \times B)$ on $(X, \mathcal{B})$. Indeed, using (20) to substitute for each instance of the term $P(\{\mathfrak{h}\})$ in both (17) and (19) gives

$$q_G(J) = \sum\nolimits_{\mathfrak{h}\in G} \xi(J_{\mathfrak{h}}|\mathfrak{h})\, \mathrm{tr}(\rho\,\Pi_{[\{\mathfrak{h}\}]}) = \mathrm{tr}[\rho\, Q_G(J)]. \tag{21}$$

Putting $J = G \times B$ in (21) and then using (18), we obtain

$$\xi^*_G(B) = \sum\nolimits_{\mathfrak{h}\in G} \xi(B|\mathfrak{h})\, \mathrm{tr}(\rho\,\Pi_{[\{\mathfrak{h}\}]}) = \mathrm{tr}[\rho\, \Xi_G(B)]. \tag{22}$$

In particular, given any experiment $e \in E$ that gives rise to the orthonormal set $G_e \subset \mathcal{S}$, the expected likelihood $\xi^*_{G_e}(B)$ of any Borel set $B \subseteq X$ is determined as the trace of the fixed density operator ρ — the quantum equivalent of a Bayesian prior — multiplied by the appropriate POVM likelihood operator Ξ_{G_e}. The quantum counterpart of equation (6) is then the roulette lottery

$$Y \ni y \mapsto \lambda^*_e(a; y) := \int_X \lambda(a, x; y)\; \mathrm{tr}[\rho\, \Xi_{G_e}(dx)] = \mathrm{tr}\left[\rho \int_X \lambda(a, x; y)\, \Xi_{G_e}(dx)\right] \tag{23}$$

whose expected utility is given by (7).

4 Concluding Summary and Research Agenda

In our laboratory game, depending on the experiment $e \in E$, player $\mathbf{E}$ implicitly selects an orthonormal subset G_e of a complex Hilbert space $\mathcal{H}$, along with a range of possible subjective likelihood laws defined on G_e. When the trace rule (20) holds, then regardless of what the set G_e may be, quantum consistency implies that there exists a "quantum state" in the form of a density operator ρ on $\mathcal{H}$ which, just as quantum Bayesians claim it should, completely characterizes player $\mathbf{D}$'s prior subjective beliefs over the unit sphere $\mathcal{S}$ of $\mathcal{H}$. Furthermore, the quantum state can be combined with a subjective likelihood operator through the trace rule in order to derive the joint distribution of states and experimental observations, as well as the marginal distribution of observations alone.

One important technical task is to extend the analysis to non-separable Hilbert spaces, which will allow likelihood operators that go beyond a discrete or pure point spectrum. In addition, future work should consider extensive form laboratory games that allow sequences of observations and/or decisions to be made at different times. Modelling these will involve solving Schrödinger's wave equation, which is most easily done in what is otherwise the rather counter-intuitive complex Hilbert space that quantum theory always uses. Not least, applying Bayesian decision theory to such extensive games may also allow further illumination of important interpretational issues such as how Bayesian updating of density operators relates to "collapsing" quantum states, as well as its relationship to the vexing measurement problem.

For the moment we have merely enriched the standard Kolmogorov probability framework to allow multiple measurable subspaces, generally incompatible, in which Bayesian rational decision making under quantum uncertainty can be described. We did without exotic probabilities whose values could not only be negative, but even involve both complex numbers and multi-dimensional positive operators. And without any special "quantum logic" that differs from ordinary set theory.

References

Anscombe, F.J. and R.J. Aumann (1963) "A Definition of Subjective Probability," *Annals of Mathematical Statistics*, **34**, 199–205.

Arrow, K.J. (1971) *Essays in the Theory of Risk Bearing* (Amsterdam: North Holland).

Boole, G. (1862) "On the Theory of Probabilities" *Philosophical Transactions of the Royal Society of London* 152: 225–52.

Caves, C.M., C.A. Fuchs, and R. Schack (2002) "Quantum Probabilities as Bayesian Probabilities" *Physical Review A* 65: 022305.

Danilov, V. and A. Lambert-Mogiliansky (2008) "Measurable Systems and Behavioral Sciences" *Mathematical Social Sciences* 55: 315–340.

Danilov, V.I. and A. Lambert-Mogiliansky (2010) "Expected Utility Theory Under Non-Classical Uncertainty" *Theory and Decision* 68: 25–47.

Feynman, R.P. (1951) "The Concept of Probability in Quantum Mechanics" *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (Berkeley, CA: University of California Press), 533–541.

Fishburn, P.C. (1982) *The Foundations of Expected Utility* (Dordrecht: D. Reidel).

Friedman, A. (1982) *Foundations of Modern Analysis* (New York: Dover).

Fuchs, C.A., and R. Schack (2009) "A Quantum-Bayesian Route to Quantum-State Space" preprint at `arXiv:0912.4252v1[quant-ph]`.

Gleason, A.M. (1957) "Measures on Closed Subspaces of a Hilbert Space" *Journal of Mathematics and Mechanics* 6: 885–894.

Griffiths, R.B. (2003) *Consistent Quantum Theory* (Cambridge: Cambridge University Press).

Gyntelberg, J. and F. Hansen (2009) "Subjective Expected Utility Theory with 'Small Worlds'", Discussion Paper 09-26, University of Copenhagen, Department of Economics; `http://econpapers.repec.org/RePEc:kud:kuiedp:0926`.

Hammond, P.J. (1998a) "Objective Expected Utility: A Consequentialist Perspective" in S. Barberà, P.J. Hammond and C. Seidl (eds.) *Handbook of Utility Theory, Vol. 1: Principles* (Dordrecht: Kluwer Academic) ch. 5, pp. 143–211.

Hammond, P.J. (1998b) "Subjective Expected Utility" in S. Barberà, P.J. Hammond and C. Seidl (eds.) *Handbook of Utility Theory, Vol. 1: Principles* (Dordrecht: Kluwer Academic) ch. 6, pp. 213–271.

Hess, K. and W. Philipp (2005) "Bell's Theorem: Critique of Proofs with and without Inequalities" in *Proceedings of the Conference on the Foundations of Probability and Physics-3* 750: 150–155 (Melville, NY: American Institute of Physics).

Janssens, B. (2004) *Quantum Measurement: A Coherent Description* Master's Thesis, Radboud University, Nijmegen; `arXiv:quant-ph/0503009v1`.

Khrennikov, A.Yu. (2008) "Contextual Probabilistic Analysis of Bell's Inequality: Nonlocality, 'Death of Reality' or Non-Kolmogorovness?" in *Proceedings of the Second International Conference on Quantum, Nano and Micro Technologies* (ICQNM) (IEEE Computer Society: Washington, DC).

La Mura, P. (2009) "Projective Expected Utility: A Subjective Formulation" *Journal of Mathematical Psychology* 53: 408–414.

Lehrer, E. and E. Shmaya (2006) "A Qualitative Approach to Quantum Probability" *Proceedings of the Royal Society A* 462: 2331–2344.

Parthasarathy, K.R. (1992) *An Introduction to Quantum Stochastic Calculus* (Basel: Birkhäuser Verlag).

Pitowsky, I. (1994) "George Boole's 'Conditions of Possible Experience' and the Quantum Puzzle" *British Journal for the Philosophy of Science* 45: 95–125.

Pitowsky, I. (2003) "Betting on the Outcomes of Measurements: A Bayesian Theory of Quantum Probability" *Studies in History and Philosophy of Modern Physics* 34: 395–414.

Savage, L.J. (1954, 1972) *Foundations of Statistics* (New York: John Wiley; and New York: Dover Publications).

Schack, R., T.A. Brun and C.M. Caves (2001) "Quantum Bayes Rule" *Physical Review A* 64: 014305.

Shafer, G. and V. Vovk (2001) *Probability and Finance: It's Only a Game!* (New York: John Wiley).

Slavnov, D.A. (2001) "Statistical Algebraic Approach to Quantum Mechanics" *Theoretical and Mathematical Physics* 129: 1385–1397.

Suppes, P. (1961) "Probability Concepts in Quantum Mechanics" *Philosophy of Science* 28: 378–389.

Suppes, P. (ed.) (1976) *Logic and Probability in Quantum Mechanics* (Dordrecht: D. Reidel).

Von Neumann, J. and O. Morgenstern (1944; 3rd edn. 1953) *Theory of Games and Economic Behavior* (Princeton: Princeton University Press).

Vorob′ev, N.N. (1962) "Consistent Families of Measures and Their Extensions" *Theory of Probability and its Applications* 7: 147–163.