Artificial Intelligence and Law manuscript No.
(will be inserted by the editor)

Proof With and Without Probabilities

Correct Evidential Reasoning with Presumptive Arguments, Coherent Hypotheses and Degrees of Uncertainty

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Received: date / Accepted: date

Abstract Evidential reasoning is hard, and errors can lead to miscarriages of justice with serious consequences. Analytic methods for the correct handling of evidence come in different styles, typically focusing on one of three tools: arguments, scenarios or probabilities. Recent research used Bayesian Networks for connecting arguments, scenarios, and probabilities. Well-known issues with Bayesian Networks were encountered: More numbers are needed than are available, and there is a risk of misinterpretation of the graph underlying the Bayesian Network, for instance as a causal model. The formalism presented here models presumptive arguments about coherent hypotheses that are compared in terms of their strength. No choice is needed between qualitative or quantitative analytic styles, since the formalism can be interpreted with and without numbers. The formalism is applied to key concepts in argumentative, scenario and probabilistic analyses of evidential reasoning, and is illustrated with a fictional crime investigation example based on Alfred Hitchcock's film 'To Catch A Thief'.

Keywords Evidential reasoning · Argumentation · Scenarios · Probabilistic reasoning · Bayesian networks · Forensic science

1 Introduction

Establishing what has happened in a crime is often not a simple task. Many errors can be made, with confirmation bias and statistical reasoning errors among the well-documented sources of mistakes (cf. also Kahneman, 2011). Recently the number of erroneous convictions in criminal trials in the Netherlands was estimated to be in the order of five to ten percent (Derksen, 2016). As a result, there is a need for analytic tools that can help prevent mistakes.

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In the literature on correct evidential reasoning, three structured analytic tools are distinguished: arguments, scenarios and probabilities (Anderson et al., 2005; Dawid et al., 2011; Kaptein et al., 2009). These tools are aimed at helping organize and structure the task of evidential reasoning, thereby supporting that good conclusions are arrived at, and foreseeable mistakes are prevented.

In an *argumentative analysis*, a structured constellation of evidence, reasons and hypotheses is considered. Typically the evidence gives rise to reasons for and against the possible conclusions considered. An argumentative analysis helps the handling of such conflicts. The early twentieth century evidence scholar John Henry Wigmore is a pioneer of argumentative analyses; cf. his famous evidence charts (Wigmore, 1913).

In a *scenario analysis*, different hypothetical scenarios about what has happened are considered side by side, and considered in light of the evidence. A scenario analysis helps the coherent interpretation of all evidence. Scenario analyses were the basis of legal psychology research about correct reasoning with evidence (Bennett and Feldman, 1981; Pennington and Hastie, 1993; Wagenaar et al., 1993).

In a *probabilistic analysis*, it is made explicit how the probabilities of the evidence and events are related. A probabilistic analysis emphasises the various degrees of uncertainty encountered in evidential reasoning, ranging from very uncertain to very certain. Probabilistic analyses of criminal evidence go back to early forensic science in the late nineteenth century (Taroni et al., 1998) and have become prominent by the statistics related to DNA profiling.

In a Netherlands-based research project,[1] artificial intelligence techniques have been used to study connections between these three tools (Verheij et al., 2016). This has resulted in the following outcomes:

- A method to manually design a Bayesian Network incorporating hypothetical scenarios and the available evidence (Vlek, 2016; Vlek et al., 2014);
- A case study testing the design method (Vlek, 2016; Vlek et al., 2014);
- A method to generate a structured explanatory text of a Bayesian Network modeled according to this method (Vlek, 2016; Vlek et al., 2016);
- An algorithm to extract argumentative information from a Bayesian Network modeling hypotheses and evidence (Timmer, 2017; Timmer et al., 2016);
- A method to incorporate argument schemes in a Bayesian Network (Timmer, 2017; Timmer et al., 2015a).

Building on earlier work in this direction (Fenton et al., 2013; Hepler et al., 2007), these results show that Bayesian Networks can be used to model arguments and structured hypotheses. Also two well-known issues encountered when using Bayesian Networks come to light:

- A Bayesian Network model typically requires many more numbers than are reasonably available;

[1] See `http://www.ai.rug.nl/~verheij/nwofs/`.

– The graph model of a Bayesian Network is formally well-defined, but there is the risk of misinterpretation, for instance unwarranted causal interpretation (Dawid, 2010) (see also Pearl, 2009).

Research has started on addressing these issues by developing an argumentation theory that connects presumptive arguments, coherent hypotheses and degrees of uncertainty (Verheij, 2014a,b; Verheij et al., 2016).

A key issue addressed in this paper is how to find an appropriate balance between qualitative and quantitative modeling styles. Building on ideas presented semi-formally by Verheij (2014b), in the present paper, a formalism is proposed in which presumptive arguments about coherent hypotheses can be compared in terms of their strengths. The formalism allows for a qualitative and a quantitative interpretation. The qualitative interpretation uses total preorders, and the quantitative interpretation probability distributions.

Key concepts used in argumentative, scenario and probabilistic analyses of reasoning with evidence are discussed in terms of the proposed formalism. The idea underlying this theoretical contribution is informally explained in the next section. The crime story of Alfred Hitchcock's famous film 'To Catch A Thief', featuring Cary Grant and Grace Kelly (1955) is used as an illustration.

2 General idea

The argumentation theory developed in this paper considers arguments that can be presumptive (also called ampliative), in the sense of logically going beyond their premises. Against the background of classical logic, an argument from premises P to conclusions Q goes beyond its premises when Q is not logically implied by P. Many arguments used in practice are presumptive. For instance, the prosecution may argue that a suspect was at the crime scene on the basis of a witness testimony. The fact that the witness has testified as such does not logically imply the fact that the suspect was at the crime scene. In particular, when the witness testimony is intentionally false, based on inaccurate observations or inaccurately remembered, the suspect may not have been at the crime scene at all. Denoting the witness testimony by P and the suspect being at the crime scene as Q, the argument from P to Q is presumptive since P does not logically imply Q. For presumptive arguments, it is helpful to consider the *case made by the argument*, defined as the conjunction of the premises and conclusions of the argument (Verheij, 2010, 2012). The case made by the argument from P to Q is $P \wedge Q$, using the conjunction of classical logic. An example of a non-presumptive argument goes from $P \wedge Q$ to Q. Here Q is logically implied by $P \wedge Q$. Presumptive arguments are often defeasible (Pollock, 1987; Toulmin, 1958), in the sense that extending the premises may lead to the retraction of conclusions.

In Figure 1, on the left, we see an argument from premises P to conclusions Q. The argument is attacked by a counterargument: the negation of Q,

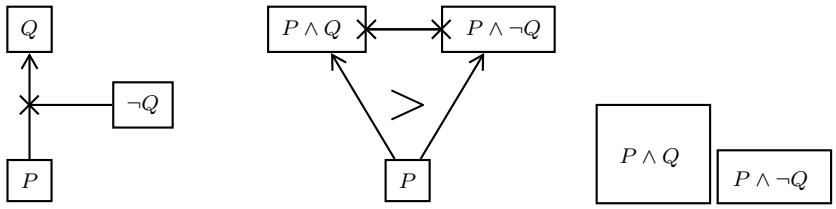

Fig. 1 General idea: an argument with a counterargument (left); arguments for conflicting cases and their comparison (middle); cases and their comparitive value (right)

denoted $\neg Q$. The case made by the argument from P to Q is $P \wedge Q$. By considering the argument from P to the case made $P \wedge Q$, the argument's presumptive character as going beyond the premises is emphasised (Figure 1, middle). An argument from P to $\neg Q$ makes the case $P \wedge \neg Q$. The two arguments from P to $P \wedge Q$ and to $P \wedge \neg Q$ are conflicting and make mutually incompatible cases. When the argument from P to $P \wedge Q$ is stronger than the argument to $P \wedge \neg Q$, the conflict is resolved, and leads to the presumptive conclusion Q. The relative strength is indicated in the figure using a $>$-sign. The relative strength of these arguments corresponds to a comparative value of the two cases $P \wedge Q$ and $P \wedge \neg Q$ being made, as suggested by the size of the corresponding boxes in the figure (Figure 1, right).

The three representations in the figure can each represent the information that Q follows presumptively from P, but not when also $\neg Q$. On the left, this is indicated by the argument from P to Q with counterargument $\neg Q$. In the middle, this is indicated by the two presumptive arguments from P making the cases $P \wedge Q$ and $P \wedge \neg Q$, where the former argument is stronger. Assuming both P and $\neg Q$, there is no conflict of arguments. On the right, this is indicated by considering that P follows from both cases, but one has a stronger relative value. Assuming both P and $\neg Q$, only one of the cases remains, viz. $P \wedge \neg Q$. In a sense, $P \wedge Q$ represents the normal case (given P) and $P \wedge \neg Q$ the exceptional one.

In Figure 1, no numbers appear. The comparison of the arguments uses the ordering relation associated with their relative strengths, indicated by the $>$-sign (in the middle). Such an ordering relation can be derived from or interpreted in a numeric representation. Figure 2 shows the numeric strengths $s(P, Q)$ and $s(P, \neg Q)$ of the middle arguments, the former larger than the latter:

$$s(P, Q) > s(P, \neg Q)$$

We discuss below that the numeric strengths $s(P, Q)$ and $s(P, \neg Q)$ can be derived from a probability function Pr, by treating strengths as conditional probabilities $\mathrm{Pr}(Q \mid P)$ and $\mathrm{Pr}(\neg Q \mid P)$. The comparison of the values of the corresponding cases $P \wedge Q$ and $P \wedge \neg Q$ is equivalently derived from the comparison of the $\mathrm{Pr}(P \wedge Q)$ and $\mathrm{Pr}(P \wedge \neg Q)$.

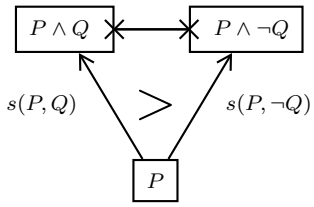

Fig. 2 Comparing argument strengths

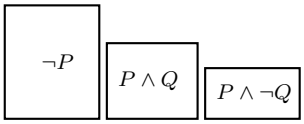

Fig. 3 Example of a case model

3 Formalism and properties

The formalism uses a classical logical language L generated from a set of propositional constants in a standard way. We write $\neg$ for negation, $\wedge$ for conjunction, $\vee$ for disjunction, $\leftrightarrow$ for equivalence, $\top$ for a tautology, and $\bot$ for a contradiction. The associated classical, deductive, monotonic consequence relation is denoted $\models$. We assume a finitely generated language, i.e., a language generated using a finite set of propositional constants.

First we define case models, formalizing the idea of cases and their preferences. Cases in a case model must be logically consistent, mutually incompatible and different. Cases are logically consistent as sentences of the classical language L. Cases are mutually incompatible, in the sense that the conjunction of case sentences that are not logically equivalent, is inconsistent. Cases are different in the sense that the set of case sentences cannot contain two elements that are logically equivalent. The comparison relation must be total and transitive (hence is what is called a total preorder, commonly modeling preference relations; Roberts, 1985).

Definition 1 (Case models) A *case model* is a pair $(C, \geq)$ with finite $C \subseteq L$, such that the following hold, for all φ, ψ and $\chi \in C$:

1. $\not\models \neg\varphi$;
2. If $\not\models \varphi \leftrightarrow \psi$, then $\models \neg(\varphi \wedge \psi)$;
3. If $\models \varphi \leftrightarrow \psi$, then $\varphi = \psi$;
4. $\varphi \geq \psi$ or $\psi \geq \varphi$;
5. If $\varphi \geq \psi$ and $\psi \geq \chi$, then $\varphi \geq \chi$.

The strict weak order $>$ standardly associated with a total preorder $\geq$ is defined as $\varphi > \psi$ if and only if it is not the case that $\psi \geq \varphi$ (for φ and $\psi \in C$). When $\varphi > \psi$, we say that φ is (strictly) preferred to ψ. The associated equivalence relation $\sim$ is defined as $\varphi \sim \psi$ if and only if $\varphi \geq \psi$ and $\psi \geq \varphi$.

Example. Figure 3 shows a case model with cases $\neg P$, $P \wedge Q$ and $P \wedge \neg Q$. $\neg P$ is (strictly) preferred to $P \wedge Q$, which in turn is preferred to $P \wedge \neg Q$.

Although the preference relations of case models are qualitative, they correspond to the relations that can be represented by real-valued functions.

Corollary 1 *Let $C \subseteq L$ be finite with elements that are logically consistent, mutually incompatible and different (properties 1, 2 and 3 in the definition of case models). Then the following are equivalent:*

1. *$(C, \geq)$ is a case model;*
2. *$\geq$ is numerically representable, i.e., there is a real valued function v on C such that for all φ and $\psi \in C$, $\varphi \geq \psi$ if and only if $v(\varphi) \geq v(\psi)$.*

The function v can be chosen with only positive values, or even with only positive integer values.

Proof. It is a standard result in order theory that total preorders on finite (or countable) sets are the ones that are representable by a real-valued function (Roberts, 1985). QED

Corollary 2 *Let $C \subseteq L$ be non-empty and finite with elements that are logically consistent, mutually incompatible and different (properties 1, 2 and 3 in the definition of case models). Then the following are equivalent:*

1. *$(C, \geq)$ is a case model;*
2. *$\geq$ is numerically representable by a probability function Pr on the algebra generated by C such that for all φ and $\psi \in C$, $\varphi \geq \psi$ if and only if $\mathrm{Pr}(\varphi) \geq \mathrm{Pr}(\psi)$.*

Proof. Pick a representing real-valued function v with only positive values as in the previous corollary, and (for elements of C) define the values of Pr as those of v divided by the sum of the v-values of all cases; then extend by summation to the algebra generated by C. When C is non-empty, Pr is a probability function on the algebra generated by C. QED

Next we define arguments. Arguments are from premises $\varphi \in L$ to conclusions $\psi \in L$.

Definition 2 (Arguments) An *argument* is a pair (φ, ψ) with φ and $\psi \in L$. The sentence φ expresses the argument's premises, the sentence ψ its conclusions, and the sentence $\varphi \wedge \psi$ the *case made* by the argument. Generalizing, a sentence $\chi \in L$ is a *premise* of the argument when $\varphi \models \chi$, a *conclusion* when $\psi \models \chi$, and a *position* in the case made by the argument when $\varphi \wedge \psi \models \chi$. An argument (φ, ψ) is *properly presumptive* when $\varphi \not\models \psi$; otherwise *non-presumptive*. An argument (φ, ψ) is a *presumption* when $\models \varphi$, i.e., when its premises are logically tautologous.

Note our use of the plural for an argument's premises, conclusions and positions. This terminological convention can be slightly confusing initially, but has been deliberately chosen since this allows us to speak of the premises $\mathtt{p}$ and $\neg\mathtt{q}$ and conclusions $\mathtt{r}$ and $\neg\mathtt{s}$ of the argument $(\mathtt{p} \wedge \neg\mathtt{q}, \mathtt{r} \wedge \neg\mathtt{s})$. Also the convention fits our non-syntactic definitions, where for instance an argument with premise χ also has logically equivalent sentences such as $\neg\neg\chi$ as a premise.

Note that we define presumptions as a specific kind of arguments, viz. from tautologous premises, and not as propositions. We have done so in order to emphasize that accepting a presumption is an inferential step that can be defeated. In this way, presumptions differ from premises, that are the basis of possible inferences, and not themselves the result of an inferential step. An example is the presumption of innocence that can be defeated by proof of guilt. (We formally discuss this example at the start of Section 4.1.)

We define three kinds of valid arguments: coherent arguments, conclusive arguments and presumptively valid arguments. A coherent argument is defined as an argument that makes a case logically implied by a case in the case model. A conclusive argument is a coherent argument, for which all cases in the case model that imply the argument's premises also imply the conclusions.

Definition 3 (Coherent and conclusive arguments) Let $(C, \geq)$ be a case model. Then we define, for all φ and $\psi \in L$:

$(C, \geq) \models (\varphi, \psi)$ if and only if $\exists \omega \in C$: $\omega \models \varphi \wedge \psi$.

We then say that the argument from φ to ψ is *coherent* with respect to the case model. We define, for all φ and $\psi \in L$:

$(C, \geq) \models \varphi \Rightarrow \psi$ if and only if $\exists \omega \in C$: $\omega \models \varphi \wedge \psi$ and $\forall \omega \in C$: if $\omega \models \varphi$, then $\omega \models \varphi \wedge \psi$.

We then say that the argument from φ to ψ is *conclusive* with respect to the case model.

Example (continued). In the case model of Figure 3, the arguments from $\top$ to $\neg P$ and to P, and from P to Q and to $\neg Q$ are coherent and not conclusive in the sense of this definition. Denoting the case model as $(C, \geq)$, we have $(C, \geq) \models (\top, \neg P)$, $(C, \geq) \models (\top, P)$, $(C, \geq) \models (P, Q)$ and $(C, \geq) \models (P, \neg Q)$. The arguments from a case (in the case model) to itself, such as from $\neg P$ to $\neg P$, or from $P \wedge Q$ to $P \wedge Q$ are conclusive. The argument $(P \vee R, P)$ is also conclusive in this case model, since all $P \vee R$-cases are P-cases. Similarly, $(P \vee R, P \vee S)$ is conclusive.

The notion of presumptive validity considered here is based on the idea that some arguments make a better case than other arguments from the same premises. More precisely, an argument is presumptively valid if there is a case in the case model implying the case made by the argument that is at least as preferred as all cases implying the premises.

Definition 4 (Presumptively valid arguments) Let $(C, \geq)$ be a case model. Then we define, for all φ and $\psi \in L$:

$(C, \geq) \models \varphi \rightsquigarrow \psi$ if and only if $\exists \omega \in C$:
1. $\omega \models \varphi \wedge \psi$; and
2. $\forall \omega' \in C$: if $\omega' \models \varphi$, then $\omega \geq \omega'$.

We then say that the argument from φ to ψ is *presumptively valid* with respect to the case model. A presumptively valid argument is *properly defeasible*, when it is not conclusive.

Example (continued). In the case model of Figure 3, the arguments from $\top$ to $\neg P$, and from P to Q are presumptively valid in the sense of this definition. Denoting the case model as $(C, \geq)$, we have formally that $(C, \geq) \models \top \rightsquigarrow \neg P$ and $(C, \geq) \models P \rightsquigarrow Q$. The coherent arguments from $\top$ to P and from P to $\neg Q$ are not presumptively valid in this sense.

Corollary 3 *1. Conclusive arguments are coherent, but there are case models with a coherent, yet inconclusive argument;*
2. Conclusive arguments are presumptively valid, but there are case models with a presumptively valid, yet inconclusive argument;
3. Presumptively valid arguments are coherent, but there are case models with a coherent, yet presumptively invalid argument.

The next proposition provides key logical properties of this notion of presumptive validity. Many have been studied for nonmonotonic inference relations (Kraus et al., 1990; Makinson, 1994; van Benthem, 1984). Given a case model $(C, \geq)$, we write $\varphi \mid\!\sim \psi$ for $(C, \geq) \models \varphi \rightsquigarrow \psi$. We write $C(\varphi)$ for the set $\{\omega \in C \mid \omega \models \varphi\}$, and refer to the elements of $C(\varphi)$ as φ-cases. For brevity, we abbreviate 'presumptively valid' to 'valid'.

(LE), for Logical Equivalence, expresses that in a valid argument premises and conclusions can be replaced by a classical equivalent (in the sense of $\models$).

(Cons), for Consistency, expresses that the conclusions of presumptively valid arguments must be consistent.

(Ant), for Antecedence, expresses that when certain premises validly imply a conclusion, the case made by the argument is also validly implied by these premises.

(RW), for Right Weakening, expresses that when the premises validly imply a composite conclusion also the intermediate conclusions are validly implied.

(CCM), for Conjunctive Cautious Monotony, expresses that the case made by a valid argument is still validly implied when an intermediate conclusion is added to the argument's premises.

(CCT), for Conjunctive Cumulative Transitivity, is a variation of the related property Cumulative Transitivity property (CT, also known as Cut). (CT)—extensively studied in the literature—has $\varphi \mid\!\sim \chi$ instead of $\varphi \mid\!\sim \psi \wedge \chi$ as a consequent. The variation is essential in our setting where the (And) property is absent (If $\varphi \mid\!\sim \psi$ and $\varphi \mid\!\sim \chi$, then $\varphi \mid\!\sim \psi \wedge \chi$). Assuming (Ant), (CCT) expresses the validity of chaining valid implication from φ via the case made in the first step $\varphi \wedge \psi$ to the case made in the second step $\varphi \wedge \psi \wedge \chi$. (See Verheij, 2010, 2012, introducing (CCT).)

Proposition 1 *Let $(C, \geq)$ be a case model. For all φ, ψ and $\chi \in L$:*

(LE) *If $\varphi \mid\!\sim \psi$, $\models \varphi \leftrightarrow \varphi'$ and $\models \psi \leftrightarrow \psi'$, then $\varphi' \mid\!\sim \psi'$.*
(Cons) $\varphi \not\mid\!\sim \bot$.

(Ant) *If* $\varphi \mathrel{|\!\sim} \psi$*, then* $\varphi \mathrel{|\!\sim} \varphi \wedge \psi$*.*
(RW) *If* $\varphi \mathrel{|\!\sim} \psi \wedge \chi$*, then* $\varphi \mathrel{|\!\sim} \psi$*.*
(CCM) *If* $\varphi \mathrel{|\!\sim} \psi \wedge \chi$*, then* $\varphi \wedge \psi \mathrel{|\!\sim} \chi$*.*
(CCT) *If* $\varphi \mathrel{|\!\sim} \psi$ *and* $\varphi \wedge \psi \mathrel{|\!\sim} \chi$*, then* $\varphi \mathrel{|\!\sim} \psi \wedge \chi$*.*

Proof. (LE): Direct from the definition. (Cons): Otherwise there would be an inconsistent element of C, contradicting the definition of a case model. (Ant): When $\varphi \mathrel{|\!\sim} \psi$, there is an ω with $\omega \models \varphi \wedge \psi$ that is $\geq$-maximal in $C(\varphi)$. Then also $\omega \models \varphi \wedge \varphi \wedge \psi$, hence $\varphi \mathrel{|\!\sim} \varphi \wedge \psi$. (RW): When $\varphi \mathrel{|\!\sim} \psi \wedge \chi$, there is an $\omega \in C$ with $\omega \models \varphi \wedge \psi \wedge \chi$ that is maximal in $C(\varphi)$. Since then also $\omega \models \varphi \wedge \psi$, we find $\varphi \mathrel{|\!\sim} \psi$. (CCM): By the assumption, we have an $\omega \in C$ with $\omega \models \varphi \wedge \psi \wedge \chi$ that is maximal in $C(\varphi)$. Since $C(\varphi \wedge \psi) \subseteq C(\varphi)$, ω is also maximal in $C(\varphi \wedge \psi)$, and we find $\varphi \wedge \psi \mathrel{|\!\sim} \chi$. (CCT): Assuming $\varphi \mathrel{|\!\sim} \psi$, there is an $\omega \in C$ with $\omega \models \varphi \wedge \psi$, maximal in $C(\varphi)$. Assuming also $\varphi \wedge \psi \mathrel{|\!\sim} \chi$, there is an $\omega' \in C$ with $\omega \models \varphi \wedge \psi \wedge \chi$, maximal in $C(\varphi \wedge \psi)$. Since $\omega \in C(\varphi \wedge \psi)$, we find $\omega' \geq \omega$. By transitivity of $\geq$, and the maximality of ω in $C(\varphi)$, we therefore have that ω' is maximal in $C(\varphi)$. As a result, $\varphi \mathrel{|\!\sim} \psi \wedge \chi$. QED

We say that an argument (φ, ψ) has coherent premises when the argument (φ, φ) from the premises to themselves is coherent. The following proposition provides some equivalent characterizations of coherent premises.

Proposition 2 *Let* $(C, \geq)$ *be a case model. The following are equivalent, for all* $\varphi \in L$*:*

1. $\varphi \mathrel{|\!\sim} \varphi$*, i.e., the argument* (φ, φ) *is presumptively valid;*
2. $\exists \omega \in C : \omega \models \varphi$ *and* $\forall \omega' \in C$*: If* $\omega' \models \varphi$*, then* $\omega \geq \omega'$*;*
3. $\exists \omega \in C : \varphi \mathrel{|\!\sim} \omega$*.*
4. $\exists \omega \in C : \omega \models \varphi$*, i.e., the argument* (φ, φ) *is coherent.*

Proof. 1 and 2 are equivalent by the definition of $\mathrel{|\!\sim}$. Assume 2. Then there is a $\geq$-maximal element ω of $C(\varphi)$. By the definition of $\mathrel{|\!\sim}$, then $\varphi \mathrel{|\!\sim} \omega$; proving 3. Assume 3. Then there is a $\geq$-maximal element ω' of $C(\varphi)$ with $\omega' \models \varphi \wedge \omega$. For this ω' also $\omega' \models \varphi$, showing 2. 4 logically follows from 2. 4 implies 2 since L is a language that generated by finitely many propositional constants. QED

Corollary 4 *Let* $(C, \geq)$ *be a case model. Then all coherent arguments have coherent premises and all presumptively valid arguments have coherent premises.*

We saw that, in the present approach, premises are coherent when they are logically implied by a case in the case model. As a result, generalisations of coherent premises are again coherent; cf. the following corollary.

Corollary 5 *Let* $(C, \geq)$ *be a case model. Then:*

If $\varphi \mathrel{|\!\sim} \varphi$ *and* $\varphi \models \psi$*, then* $\psi \mathrel{|\!\sim} \psi$*.*

We now consider some properties that use a subset L^* of the language L. The set L^* consists of the logical combinations of the cases of the case model using negation, conjunction and logical equivalence (cf. the algebra underlying

probability functions (Roberts, 1985)). L^* is the set of *case expressions* associated with a case model.

(Coh), for Coherence, expresses that coherent premises correspond to a consistent case expression implying the premises. (Ch), for Choice, expresses that, given two coherent case expressions, at least one of three options follows validly: the conjunction of the case expression, or the conjunction of one of them with the negation of the other. (OC), for Ordered Choice, expresses that preferred choices between case expressions are transitive. Here we say that a case expression is a *preferred choice* over another, when the former follows validly from the disjunction of both.

Definition 5 (Preferred cases) Let $(C, \geq)$ be a case model, $\varphi \in L$, and $\omega \in C$. Then ω expresses a *preferred case* of φ if and only if $\varphi \mathrel{|\!\sim} \omega$.

Proposition 3 *Let $(C, \geq)$ be a case model, and $L^* \subseteq L$ the closure of C under negation, conjunction and logical equivalence. Writing $\mathrel{|\!\sim}^*$ for the restriction of $\mathrel{|\!\sim}$ to L^*, we have, for all φ, ψ and $\chi \in L^*$:*

(Coh) $\varphi \mathrel{|\!\sim} \varphi$ *if and only if* $\exists \varphi^* \in L^*$ *with* $\varphi^* \not\models \bot$ *and* $\varphi^* \models \varphi$*;*
(Ch) *If* $\varphi \mathrel{|\!\sim}^* \varphi$ *and* $\psi \mathrel{|\!\sim}^* \psi$*, then* $\varphi \vee \psi \mathrel{|\!\sim}^* \neg\varphi \wedge \psi$ *or* $\varphi \vee \psi \mathrel{|\!\sim}^* \varphi \wedge \psi$ *or* $\varphi \vee \psi \mathrel{|\!\sim}^* \varphi \wedge \neg\psi$*;*
(OC) *If* $\varphi \vee \psi \mathrel{|\!\sim}^* \varphi$ *and* $\psi \vee \chi \mathrel{|\!\sim}^* \psi$*, then* $\varphi \vee \chi \mathrel{|\!\sim}^* \varphi$*.*

Proof. (Coh): By Proposition 2, $\varphi \mathrel{|\!\sim} \varphi$ if and only if there is an $\omega \in C$ with $\omega \models \varphi$. The property (Coh) follows since $C \subseteq L^*$ and, for all consistent $\varphi^* \in L^*$, there is an $\omega \in C$ with $\omega \models \varphi^*$.
(Ch): Consider sentences φ and $\psi \in L^*$ with $\varphi \mathrel{|\!\sim}^* \varphi$ and $\psi \mathrel{|\!\sim}^* \psi$. Then, by Corollary 5, $\varphi \vee \psi \mathrel{|\!\sim} \varphi \vee \psi$. By Proposition 2, there is an $\omega \in C$, with $\omega \models \varphi \vee \psi$. The sentences φ and ψ are elements of L^*, hence also the sentences $\varphi \wedge \neg\psi$, $\varphi \wedge \psi$ and $\neg\varphi \wedge \psi \in L^*$. All are logically equivalent to disjunctions of elements of C (possibly the empty disjunction, logically equivalent to $\bot$). Since $\omega \models \varphi \vee \psi$, $\models \varphi \vee \psi \leftrightarrow (\varphi \wedge \neg\psi) \vee (\varphi \wedge \psi) \vee (\neg\varphi \wedge \psi)$, and the elements of C are mutually incompatible, we have $\omega \models \varphi \wedge \neg\psi$ or $\omega \models \varphi \wedge \psi$ or $\omega \models \neg\varphi \wedge \psi$. By Proposition 2, it follows that $\varphi \vee \psi \mathrel{|\!\sim}^* \neg\varphi \wedge \psi$ or $\varphi \vee \psi \mathrel{|\!\sim}^* \varphi \wedge \psi$ or $\varphi \vee \psi \mathrel{|\!\sim}^* \varphi \wedge \neg\psi$.

(OC): By $\varphi \vee \psi \mathrel{|\!\sim}^* \varphi$, there is an $\omega \models \varphi$ maximal in $C(\varphi \vee \psi)$. By $\psi \vee \chi \mathrel{|\!\sim}^* \psi$, there is an $\omega' \models \psi$ maximal in $C(\psi \vee \chi)$. Since $\omega \models \varphi$, $\omega \in C(\varphi \vee \chi)$. Since $\omega' \models \psi$, $\omega' \in C(\varphi \vee \psi)$, hence $\omega \geq \omega'$. Hence ω is maximal in $C(\varphi \vee \chi)$, hence $\varphi \vee \chi \mathrel{|\!\sim} \varphi$. Since $\chi \in L^*$, $\varphi \vee \chi \mathrel{|\!\sim}^* \varphi$. QED

The properties in propositions 1 and 3 are the basis of qualitative and quantitative representation results for the inference relation $\mathrel{|\!\sim}$. See Verheij (2016a), also for other formal properties of the proposal. In Section 4.3, we show how the probabilistic representation of case models (Corollary 2) gives rise to probabilistic representations of our three kinds of argument validity: coherence, conclusiveness, and presumptive validity.

The history of research in Artificial Intelligence that combines arguments, hypotheses and uncertainty is extensive and varied. Without claiming a representative selection, we mention a few examples in order to position the present formalism. We already mentioned the work by Kraus et al. (1990) on a preferential semantics for non-monotonic inference. Formal differences include that the present proposal uses cases, not worlds as primitives in the semantics, and that the (And)-rule (If $\varphi \mathrel{|\!\sim} \psi$ and $\varphi \mathrel{|\!\sim} \chi$, then $\varphi \mathrel{|\!\sim} \psi \wedge \chi$.) does not hold for our notion of presumptive validity. See Verheij (2016a) for further formal information. Non-formal differences are that the present proposal is designed to be a balance between qualitative and quantitative modeling, and has been applied to the modeling of evidential reasoning (this paper) and normative reasoning (Verheij, 2016c). Kohlas et al. (1998) proposes a probabilistic approach to model-based diagnostics using arguments supporting hypotheses about the state of a system. It is discussed that numerical degrees of support can be looked at as conditional probabilities. Dung and Thang (2010) defines probabilistic adaptations of abstract and assumption-based argumentation. Hunter (2013) studies probability distributions in the settings of abstract and logical argumentation, leading to an analysis of different kinds of inconsistency that can arise. Benferhat et al. (2000) study non-monotonic reasoning in terms of default reasoning, building on Adams' epsilon semantics in terms of extreme probabilities. Fagin and Halpern (1994) study reasoning about knowledge and probability, studying a language that allows for the explicit mentioning of an agent's numeric probabilistic beliefs. Satoh (1990) studies non-monotonic reasoning with a probabilistic semantics such that new information only leads to non-monotonicity when it is contradicting previous information.

4 A formal analysis of some key concepts

We now use the formalism of case models and presumptive validity above for a discussion of some key concepts associated with the argumentative, scenario and probabilistic analysis of evidential reasoning.

4.1 Arguments

In an argumentative analysis, it is natural to classify arguments with respect to the nature of the support their premises give their conclusions. We already defined non-presumptive and presumptive arguments (Definition 2), and—with respect to a case model—presumptively valid and properly defeasible arguments (Definition 4). We illustrate these notions in an example about the presumption of innocence.

Let $\mathtt{inn}$ denote that a suspect is innocent, and $\mathtt{gui}$ that he is guilty. Then the argument $(\mathtt{inn}, \neg\mathtt{gui})$ is properly presumptive, since $\mathtt{inn} \not\models \neg\mathtt{gui}$. The argument $(\mathtt{inn} \wedge \neg\mathtt{gui}, \neg\mathtt{gui})$ is non-presumptive, since $\mathtt{inn} \wedge \neg\mathtt{gui} \models \neg\mathtt{gui}$.

Fig. 4 A case model for presumption

Presumptive validity and defeasibility are illustrated using a case model. Consider the case model with two cases $\texttt{inn} \wedge \neg\texttt{gui}$ and $\neg\texttt{inn} \wedge \texttt{gui} \wedge \texttt{evi}$ with the first case preferred to the second (Figure 4; the size of the cases' rectangles measures their preference). Here $\texttt{evi}$ denotes evidence for the suspect's guilt. Then the properly presumptive argument $(\texttt{inn}, \neg\texttt{gui})$ is presumptively valid with respect to this case model since the conclusion $\neg\texttt{gui}$ follows in the case $\texttt{inn} \wedge \neg\texttt{gui}$ that is a preferred case of the premise $\texttt{inn}$. The argument is conclusive since there are no other cases implying $\texttt{inn}$. The argument $(\top, \texttt{inn})$—in fact a presumption now that its premises are tautologous—is presumptively valid since $\texttt{inn}$ follows in the preferred case $\texttt{inn} \wedge \neg\texttt{gui}$. This shows that the example represents what is called the presumption of innocence, when there is no evidence. This argument is properly defeasible since in the other case of the argument's premises the conclusion does not follow. In fact, the argument $(\texttt{evi}, \texttt{inn})$ is not coherent since there is no case in which both $\texttt{evi}$ and $\texttt{inn}$ follow. The argument $(\texttt{evi}, \texttt{gui})$ is presumptively valid, even conclusive.

In argumentative analyses, different kinds of argument attack are considered. John Pollock made the famous distinction between two kinds of—what he called—argument defeaters (Pollock, 1987, 1995). A rebutting defeater is a reason for a conclusion that is the opposite of the conclusion of the attacked argument, whereas an undercutting defeater is a reason that attacks not the conclusion itself, but the connection between reason and conclusion. Joseph Raz made a related famous distinction of exclusionary reasons that always prevail, independent of the strength of competing reasons (Raz, 1990) (see also Richardson, 2013).

Unlike in the work of Pollock, in the present proposal, undercutting and rebutting attack are not treated as separate primitives. Instead they are specializations of a general idea of attack defined in terms of case models. In this connection, Figure 1 can be confusing as the graphical representation of the argument and counterargument (in the figure on the left) suggests that $\neg Q$ attacks the connection between P and Q, much like an undercutter. But the attack consists in the negation of the conclusion Q of the argument from P, reminiscent of a rebutter. We show how the distinction between undercutting and rebutting attack can still be made in the present proposal.

We propose the following terminology.

Definition 6 (Defeating circumstances) Let $(C, \geq)$ be a case model, and (φ, ψ) a presumptively valid argument. Then circumstances χ are *defeating* or *successfully attacking* when $(\varphi \wedge \chi, \psi)$ is not presumptively valid. Defeating cir-

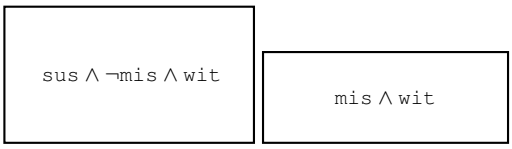

Fig. 5 A case model for undercutting

cumstances are *rebutting* when $(\varphi \wedge \chi, \neg\psi)$ is presumptively valid; otherwise they are *undercutting*. Defeating circumstances are *excluding* when $(\varphi \wedge \chi, \psi)$ is not coherent.

Continuing the example of the case model illustrated in Figure 4, we find the following. The circumstances $\mathtt{evi}$ defeat the presumptively valid argument $(\top, \mathtt{inn})$ since $(\mathtt{evi}, \mathtt{inn})$ is not presumptively valid. In fact, these circumstances are excluding since $(\mathtt{evi}, \mathtt{inn})$ is not coherent. The circumstances are also rebutting since the argument for the opposite conclusion $(\mathtt{evi}, \neg\mathtt{inn})$ is presumptively valid. Note that this example of rebutting defeat is defeat of a presumption (in the sense of Definition 2), hence can be regarded as a formalization of the idea of undermining defeat that is the basis of argumentation formalisms in which defeat is assumption-based (Bondarenko et al., 1997; Verheij, 2003). See also the discussion of arguments with prima facie assumptions by van Eemeren et al. (2014).

Undercutting can be illustrated with an example about a lying witness. Consider a case model with these two cases:

Case 1: $\mathtt{sus} \wedge \neg\mathtt{mis} \wedge \mathtt{wit}$
Case 2: $\mathtt{mis} \wedge \mathtt{wit}$

In the cases, there is a witness testimony ($\mathtt{wit}$) that the suspect was at the crime scene ($\mathtt{sus}$). In Case 1, the witness was not misguided ($\neg\mathtt{mis}$), in Case 2 he was. In Case 1, the suspect was indeed at the crime scene; in Case 2, the witness was misguided and it is unspecified whether the suspect was at the crime scene or not. In the case model, Case 1 is preferred to Case 2 (Figure 5), representing that witnesses are usually not misguided.

Since Case 1 is a preferred case of $\mathtt{wit}$, the argument $(\mathtt{wit}, \mathtt{sus})$ is presumptively valid: the witness testimony provides a presumptively valid argument for the suspect having been at the crime scene. The argument's conclusion can be strengthened to include that the witness was not misguided. Formally, this is expressed by saying that $(\mathtt{wit}, \mathtt{sus} \wedge \neg\mathtt{mis})$ is a presumptively valid argument.

When the witness was misguided after all ($\mathtt{mis}$), there are circumstances defeating the argument $(\mathtt{wit}, \mathtt{sus})$. This can be seen by considering that Case 2 is the only case in which $\mathtt{wit} \wedge \mathtt{mis}$ follows, hence is preferred. Since $\mathtt{sus}$ does not follow in Case 2, the argument $(\mathtt{wit} \wedge \mathtt{mis}, \mathtt{sus})$ is not presumptively valid. The misguidedness is not rebutting, hence undercutting since $(\mathtt{wit} \wedge \mathtt{mis}, \neg\mathtt{sus})$ is not presumptively valid. The misguidedness is excluding since the argument $(\mathtt{wit} \wedge \mathtt{mis}, \mathtt{sus})$ is not even coherent.

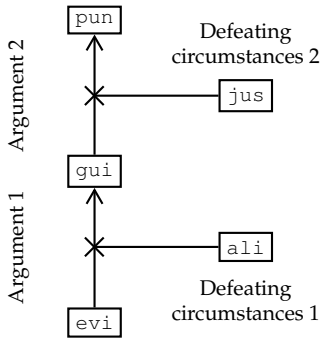

Fig. 6 Chained arguments

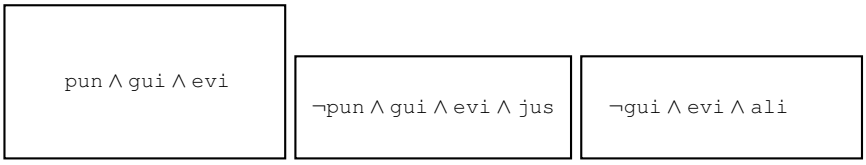

Fig. 7 Case model for chained arguments

Arguments can typically be chained, namely when the conclusion of one is a premise of another. For instance when there is evidence (evi) that a suspect is guilty of a crime (gui), the suspect's guilt can be the basis of punishing the suspect (pun). For both steps there are typical defeating circumstances. The step from the evidence to guilt is blocked when there is a solid alibi (ali), and the step from guilt to punishing is blocked when there are grounds of justification (jus), such as force majeure. Cf. Figure 6.

A case model with three cases can illustrate such chaining:

Case 1: $\mathtt{pun} \wedge \mathtt{gui} \wedge \mathtt{evi}$
Case 2: $\neg\mathtt{pun} \wedge \mathtt{gui} \wedge \mathtt{evi} \wedge \mathtt{jus}$
Case 3: $\neg\mathtt{gui} \wedge \mathtt{evi} \wedge \mathtt{ali}$

Cf. Figure 7. In the case model, Case 1 is preferred to Case 2 and Case 3, modeling that the evidence typically leads to guilt and punishing, unless there are grounds for justification (Case 2) or there is an alibi (Case 3). Cases 2 and 3 are preferentially equivalent.

In this case model, the following arguments are presumptively valid:

Argument 1 (presumptively valid): $(\mathtt{evi}, \mathtt{gui})$
Argument 2 (presumptively valid): $(\mathtt{gui}, \mathtt{pun})$
Argument 3 (presumptively valid): $(\mathtt{evi}, \mathtt{gui} \wedge \mathtt{pun})$

Arguments 1 and 3 are presumptively valid since Case 1 is the preferred case among those in which evi follows (Cases 1, 2 and 3); Argument 2 is since Case 1 is the preferred case among those in which gui follows (Cases 1 and 2).

By chaining arguments 1 and 2, the case for $\texttt{gui} \wedge \texttt{pun}$ can be based on the evidence $\texttt{evi}$ as in Argument 3.

The following arguments are not presumptively valid in this case model:

Argument 4 (not presumptively valid): $(\texttt{evi} \wedge \texttt{ali}, \texttt{gui})$
Argument 5 (not presumptively valid): $(\texttt{gui} \wedge \texttt{jus}, \texttt{pun})$

This shows that Arguments 1 and 2 are defeated by circumstances $\texttt{ali}$ and $\texttt{jus}$, respectively:

Defeating circumstances 1 (attacking Argument 1): $\texttt{ali}$
Defeating circumstances 2 (attacking Argument 2): $\texttt{jus}$

The structural relations of the arguments 1 and 2 and their defeating circumstances 1 and 2 are graphically shown in Figure 6.

As expected, chaining the arguments fails under both of these defeating circumstances, as shown by the fact that these two arguments are not presumptively valid:

Argument 6 (not presumptively valid): $(\texttt{evi} \wedge \texttt{ali}, \texttt{gui} \wedge \texttt{pun})$
Argument 7 (not presumptively valid): $(\texttt{evi} \wedge \texttt{jus}, \texttt{gui} \wedge \texttt{pun})$

But the first step of the chain—the step to guilt—can be made when there are grounds for justification. Formally, this can be seen by the presumptive validity of this argument:

Argument 8 (presumptively valid): $(\texttt{evi} \wedge \texttt{jus}, \texttt{gui})$

This example shows how the preference ordering of cases is connected to the overriding of arguments by their exceptions. Here we see that the exceptional cases about grounds of justification and alibi are less preferred than Case 1. One could say that because Case 1 is preferred the exceptional cases 2 and 3 are ignored given only $\texttt{evi}$ as a premise. The three arguments from $\texttt{evi}$ to each of the cases separately are coherent, but of these only the argument to Case 1 is presumptively valid. Since Case 3 does not logically imply the defeating circumstances, adding $\texttt{ali}$ or $\texttt{jus}$ to the premises makes Case 3 no longer coherently supported, hence certainly not presumptively valid. Cf. Arguments 6 and 7 that make a case that logically implies Case 3, but are not presumptively valid and not coherent.

4.2 Scenarios

In the literature on scenario analyses, several notions are used in order to analyze the 'quality' of the scenarios considered. Three notions are prominent: a scenario's consistency, a scenario's plausibility and a scenario's completeness (Pennington and Hastie, 1993; Wagenaar et al., 1993). In this literature, these notions are part of an informally discussed theoretical background, having prompted work in AI & Law on formalizing these notions (Bex, 2011;

Verheij and Bex, 2009; Vlek et al., 2015). A scenario is consistent when it does not contain contradictions. For instance, a suspect cannot be both at home and at the crime scene. A scenario is plausible when it fits commonsense knowledge about the world. For instance, in a murder scenario, a victim's death caused by a shooting seems a plausible possibility. A scenario is complete when all relevant elements are in the scenario. For instance, a murder scenario requires a victim, an intention and premeditation. We now propose a formal treatment of these notions using the formalism presented.

The consistency of a scenario could simply be taken to correspond to logical consistency. A stronger notion of consistency uses the world knowledge represented in a case model, and emphasises the coherence of a scenario in the sense of the present formalism. In this way, we connect to the term coherence that also appears in the literature on scenario-based evidence analysis, with various connotations.

In our proposal, some coherent scenarios fit the world knowledge represented in the case model better than others, since some are presumptively valid. We can say that a scenario is plausible (given a case model) when it is a presumptively valid conclusion of the evidence. Continuing the example, the complete scenario $\texttt{vic} \wedge \texttt{int} \wedge \texttt{pre}$ is then plausible given $\texttt{evi}$, but also subscenarios such as $\texttt{vic} \wedge \texttt{int}$ (leaving the premeditation unspecified) and $\texttt{int} \wedge \texttt{pre}$ (with no victim, only intention and premeditation). This notion of a scenario's plausibility depends on the evidence, in contrast with the mentioned literature (Pennington and Hastie, 1993; Wagenaar et al., 1993), where plausibility is treated as being independent from the evidence. The present proposal includes an evidence-independent notion of plausibility, by considering a scenario as plausible—independent of the evidence—when it is plausible given no evidence, i.e., when the scenario is a presumptively valid presumption. In the present setting, plausibility can be connected to the preference ordering on cases given the evidence, when scenarios are complete.

In the formal proposal here, besides coherence and presumptive validity, we have encountered a third notion of validity: conclusiveness. This notion can be used to represent that there is no remaining doubt about a scenario given the knowledge in the case model: the scenario is beyond a reasonable doubt. The doubt that always remains is transferred to doubt about whether everything that needs to be considered is in the case model. When the case model is the result of a process of critical, careful and open-minded scrutiny, and has been performed with appropriate effort, such remaining doubt could be dubbed 'unreasonable' (Verheij, 2014b).

We summarize the discussed definitions of coherence, completeness and reasonable doubt, each in an evidence-independent and evidence-dependent variant. Sentences σ are intended to express scenarios, sentences ϵ the evidence.

Definition 7 Let $(C, \geq)$ be a case model, and $\sigma \in L$. Then we define:

1. σ is *coherent* if and only if the argument $(\top, \sigma)$ is coherent;
2. σ is *plausible* if and only if the argument $(\top, \sigma)$ is presumptively valid;

3. σ is *beyond a reasonable doubt* if and only if the argument $(\top, \sigma)$ is conclusive.

Definition 8 Let $(C, \geq)$ be a case model, and σ and $\epsilon \in L$. Then we define:

1. σ is *coherent given* ϵ if and only if the argument (ϵ, σ) is coherent;
2. σ is *plausible given* ϵ if and only if the argument (ϵ, σ) is presumptively valid;
3. σ is *beyond a reasonable doubt given* ϵ if and only if the argument (ϵ, σ) is conclusive.

The completeness of a scenario can here be defined using a notion of maximally specific conclusions, or extensions, as follows.

Definition 9 (Extensions) Let $(C, \geq)$ be a case model, and (φ, ψ) a presumptively valid argument. Then the case made by the argument (i.e., $\varphi \wedge \psi$) is an *extension* of φ when there is no presumptively valid argument from φ that makes a case that is logically more specific.

For instance, consider a case model in which the case $\texttt{vic} \wedge \texttt{int} \wedge \texttt{pre} \wedge \texttt{evi}$ is a preferred case of $\texttt{evi}$. The case expresses a situation in which there is evidence ($\texttt{evi}$) for a typical murder: there is a victim ($\texttt{vic}$), there was the intention to kill ($\texttt{int}$), and there was premeditation ($\texttt{pre}$). In such a case model, this case is an extension of the evidence $\texttt{evi}$. A scenario can now be considered complete with respect to certain evidence when the scenario conjoined with the evidence is its own extension. In the example, the sentence $\texttt{vic} \wedge \texttt{int} \wedge \texttt{pre}$ is a complete scenario given $\texttt{evi}$ as the scenario conjoined with the evidence is its own extension. The sentence $\texttt{vic} \wedge \texttt{int}$ is not a complete scenario given $\texttt{evi}$, as the extension of $\texttt{vic} \wedge \texttt{int} \wedge \texttt{evi}$ also implies $\texttt{pre}$.

Definition 10 Let $(C, \geq)$ be a case model, and $\sigma \in L$. Then we define:

σ is *complete given* ϵ if and only if $\sigma \wedge \epsilon$ is an extension of ϵ.

In the literature, scenario schemes have been used to represent a scenario's completeness (Bex, 2011; Bex and Verheij, 2013; Verheij et al., 2016; Vlek et al., 2014, 2016), taking inspiration from the use of scripts in artificial intelligence and cognitive science (Schank and Abelson, 1977). Here the cases in a case model are used to represent completeness.

4.3 Probabilities

The literature on the probabilistic analysis of reasoning with evidence uses the probability calculus as formal background. A key formula is the well-known Bayes' theorem, stating that for events H and E the following relation between probabilities holds:

$$\Pr(H|E) = \frac{\Pr(E|H)}{\Pr(E)} \cdot \Pr(H)$$

Thinking of H as a hypothesis and E as evidence, here the posterior probability $\Pr(H|E)$ of the hypothesis given the evidence can be computed by multiplying the prior probability $\Pr(H)$ and the Bayes factor $\Pr(E|H)/\Pr(E)$.

A formula that is especially often encountered in the literature on evidential reasoning is the following odds version of Bayes' theorem:

$$\frac{\Pr(H|E)}{\Pr(\neg H|E)} = \frac{\Pr(E|H)}{\Pr(E|\neg H)} \cdot \frac{\Pr(H)}{\Pr(\neg H)}$$

Here the posterior odds $\Pr(H|E)/\Pr(\neg H|E)$ of the hypothesis given the evidence is found by multiplying the prior odds $\Pr(H)/\Pr(\neg H)$ with the likelihood ratio $\Pr(E|H)/\Pr(E|\neg H)$. This formula is important since the likelihood ratio can sometimes be estimated, for instance in the case of DNA evidence. In fact, it is a key lesson in probabilistic approaches to evidential reasoning that the evidential value of evidence, as measured by a likelihood ratio, does not by itself determine the posterior probability of the hypothesis considered. As the formula shows, the prior probability of the hypothesis is needed to determine the posterior probability given the likelihood ratio. Just as Bayes' theorem, the likelihood ratio obtains in a probabilistic realization of a case model in our sense.

In Sections 4.1 and 4.2, we focused on arguments and scenarios, which have primarily (but not exlusively) been studied using qualitative methods. Here we show that key notions of our approach can be given a quantitative, probabilistic representation. In this way, we intend to show the balanced connection between qualitative and quantitative analytic methods.

In particular, we turn to the quantitative representation of our three notions of argument validity: coherence, conclusiveness and presumptive validity. We use the probabilistic representation of case models as in Corollary 2. The representing probability functions used there are functions on the algebra generated by C. It is convenient to extend such functions to the language L.

Definition 11 Let $(C, \geq)$ be a case model (with C non-empty) represented by a probability function $\Pr$ as in Corollary 2. Then we define, for all φ and $\psi \in L$:

1. $\Pr(\varphi) := \sum_{\omega \in C \text{ and } \omega \models \varphi} Pr(\omega)$;
2. $\Pr(\psi|\varphi) := \Pr(\varphi \wedge \psi)/\Pr(\varphi)$ if $\Pr(\varphi) > 0$.

Note that the extension $\Pr$ to L only behaves exactly like the logical generalization of a probability function when restricted to sentences corresponding to the algebra generated by C. Consider for instance a language L generated by propositional constants p and q and case model $(\{p\}, \{(p, p)\})$ represented by $\Pr$. Then $\Pr(p) = 1$ and $\Pr(\neg p) = 0$, as expected in a probabilistic setting where the probabilities of complements add up to 1. However, $\Pr(q)$ and $\Pr(\neg q)$ are both equal to 0.

Proposition 4 *(Coherence, quantitative) Let $(C, \geq)$ be a case model (with C non-empty) represented by a probability function $\Pr$ as in Corollary 2. Then, for all φ and $\psi \in L$, the following are equivalent:*

1. $(C, \geq) \models (\varphi, \psi)$;
2. $\Pr(\varphi \wedge \psi) > 0$.

Proof. Immediate using the definitions. An argument (φ, ψ) is coherent if and only if there is a case ω in C from which $\varphi \wedge \psi$, the case made by the argument, follows logically. And, since $\Pr$ in Corollary 2 is positive on C, the definition of the extension of $\Pr$ to L gives that this is the case if and only if $\Pr(\varphi \wedge \psi) > 0$. QED

Proposition 5 *(Conclusiveness, quantitative) Let $(C, \geq)$ be a case model (with C non-empty) represented by a probability function $\Pr$ as in Corollary 2. Then, for all φ and $\psi \in L$, the following are equivalent:*

1. $(C, \geq) \models \varphi \Rightarrow \psi$;
2. $\Pr(\psi|\varphi) = 1$.

Proof. An argument (φ, ψ) is conclusive if and only if it is coherent and all φ-cases in C are also $\varphi \wedge \psi$-cases. This is the case if and only if $\Pr(\varphi \wedge \psi) > 0$ and $\Pr(\varphi \wedge \psi) = \Pr(\varphi)$. Since $\Pr(\varphi \wedge \psi) > 0$ implies $\Pr(\varphi) > 0$, that is equivalent to $\Pr(\psi|\varphi) = 1$. QED

Proposition 6 *(Presumptive validity, quantitative) Let $(C, \geq)$ be a case model (with C non-empty) represented by a probability function $\Pr$ as in Corollary 2. Then, for all φ and $\psi \in L$, the following are equivalent:*

1. $(C, \geq) \models \varphi \rightsquigarrow \psi$;
2. $\exists \omega \in C$:
   (a) $\omega \models \varphi \wedge \psi$; *and*
   (b) $\forall \omega' \in C$: *if* $\omega' \models \varphi$, *then* $\Pr(\omega) \geq \Pr(\omega')$;
3. $\exists \omega \in C$:
   (a) $\omega \models \varphi \wedge \psi$; *and*
   (b) $\forall \omega' \in C : \Pr(\omega \mid \varphi) \geq \Pr(\omega' \mid \varphi)$.

Proof. An argument (φ, ψ) is presumptively valid if and only there is a $\varphi \wedge \psi$-case ω that is $\geq$-maximal among the φ-cases in C. Hence the equivalence of 1 and 2. Noting that $\omega \models \varphi \wedge \psi$ implies $\Pr(\varphi \wedge \psi) > 0$, which implies $\Pr(\varphi) > 0$, which in turn implies that $\Pr(\omega'|\varphi)$ is defined for all $\omega' \in C$, we find that 2 and 3 are also equivalent. QED

The propositions show how the qualitatively defined notions of coherence, conclusiveness and presumptive validity have equivalent quantitative characterizations. For presumptive validity, one is in terms of the comparative value of cases, measured as a probability (part 2 of the proposition), the other in terms of the comparative strength of arguments, measured as a conditional probability (part 3 of the proposition).

We discuss an example, adapting our earlier treatment of the presumption of innocence. Consider a crime case where two pieces of evidence are found, one after another. In combination, they are considered to prove the suspect's

guilt beyond a reasonable doubt. For instance, one piece of evidence is a witness who claims to have seen the suspect committing the crime (evi), and a second piece of evidence is DNA evidence matching the suspect's profile (evi'). The issue is whether the suspect is innocent (inn) or guilty (gui). Consider now a case model with four cases:

Case 1: $\texttt{inn} \wedge \neg\texttt{gui} \wedge \neg\texttt{evi}$
Case 2: $\neg\texttt{inn} \wedge \texttt{gui} \wedge \texttt{evi} \wedge \neg\texttt{evi'}$
Case 3: $\texttt{inn} \wedge \neg\texttt{gui} \wedge \texttt{evi} \wedge \neg\texttt{evi'}$
Case 4: $\neg\texttt{inn} \wedge \texttt{gui} \wedge \texttt{evi} \wedge \texttt{evi'}$

Case 1 expresses the situation when no evidence has been found, hence the suspect is considered innocent and not guilty. In order to express that by default there is no evidence concerning someone's guilt, this case has highest preference. Cases 2 and 3 express the situation that the first piece of evidence is found. Case 2 expresses guilt, Case 3 innocence, still considered a possibility given only the first piece of evidence. In order to express that evi makes the suspect's guilt more plausible than his innocence, Case 2 has higher preference than Case 3. Case 4 represents the situation that both pieces of evidence are available, proving guilt. It has lowest preference. Summarizing the preference relation we have:

Case 1 > Case 2 > Case 3 > Case 4

Qualitatively, the following hold in this case model:

1. The argument $(\top, \texttt{inn})$ for innocence given no evidence is coherent, presumptively valid and not conclusive;
2. The argument $(\top, \texttt{gui})$ for guilt given no evidence is coherent, not presumptively valid and not conclusive;
3. The argument $(\texttt{evi}, \texttt{inn})$ for innocence given only the first piece of evidence is coherent, not presumptively valid and not conclusive;
4. The argument $(\texttt{evi}, \texttt{gui})$ for guilt given only the first piece of evidence is coherent, presumptively valid and not conclusive;
5. The argument $(\texttt{evi} \wedge \texttt{evi'}, \texttt{inn})$ for innocence given both the first and the second piece of evidence is not coherent, not presumptively valid and not conclusive.
6. The argument $(\texttt{evi} \wedge \texttt{evi'}, \texttt{gui})$ for guilt given both the first and the second piece of evidence is coherent, presumptively valid and conclusive.

In Tables 1 and 2, we translate these remarks to their quantitative versions using the Propositions 4, 5, and 6. Here we assume that probability function $\Pr$ represents the case model as in Corollary 2 and has been extended to a function on L as in the propositions. As expected, the specific numbers used in $\Pr$ do not matter much. It is mostly their relative sizes that count. For instance, we could use $\Pr$ with $\Pr(\text{Case 1}) = 0.4$, $\Pr(\text{Case 2}) = 0.3$, $\Pr(\text{Case 3}) = 0.2$, and $\Pr(\text{Case 4}) = 0.1$. If information about actual distributions for this example is available (for instance about the proportion of possible suspects for

Table 1 Coherence and conclusiveness of the example's arguments

Argument	Coherence	Conclusiveness
$(\top, \texttt{inn})$	yes: $\Pr(\texttt{inn}) > 0$	no: $\Pr(\texttt{inn}) < 1$
$(\top, \texttt{gui})$	yes: $\Pr(\texttt{inn}) > 0$	no: $\Pr(\texttt{inn}) < 1$
$(\texttt{evi}, \texttt{inn})$	yes: $\Pr(\texttt{inn} \wedge \texttt{evi}) > 0$	no: $\Pr(\texttt{inn} \mid \texttt{evi}) < 1$
$(\texttt{evi}, \texttt{gui})$	yes: $\Pr(\texttt{inn} \wedge \texttt{evi}) > 0$	no: $\Pr(\texttt{inn} \mid \texttt{evi}) < 1$
$(\texttt{evi} \wedge \texttt{evi'}, \texttt{inn})$	no: $\Pr(\texttt{inn} \wedge \texttt{evi} \wedge \texttt{evi'}) = 0$	no: $\Pr(\texttt{inn} \mid \texttt{evi} \wedge \texttt{evi'}) < 1$
$(\texttt{evi} \wedge \texttt{evi'}, \texttt{gui})$	yes: $\Pr(\texttt{inn} \wedge \texttt{evi} \wedge \texttt{evi'}) > 0$	yes: $\Pr(\texttt{inn} \mid \texttt{evi} \wedge \texttt{evi'}) = 1$

Table 2 Presumptive validity of the example's arguments

Argument	Presumptive validity
$(\top, \texttt{inn})$	yes: $\Pr(\text{Case 1}) > \Pr(\text{Case 2}) > \Pr(\text{Case 3}) > \Pr(\text{Case 4})$
$(\top, \texttt{gui})$	no: $\Pr(\text{Case 2}) < \Pr(\text{Case 1})$; $\Pr(\text{Case 4}) < \Pr(\text{Case 1})$
$(\texttt{evi}, \texttt{inn})$	no: $\Pr(\text{Case 3} \mid \texttt{evi}) < \Pr(\text{Case 2} \mid \texttt{evi})$
$(\texttt{evi}, \texttt{gui})$	yes: $\Pr(\text{Case 2} \mid \texttt{evi}) > \Pr(\text{Case 3} \mid \texttt{evi}) > \Pr(\text{Case 4} \mid \texttt{evi})$
$(\texttt{evi} \wedge \texttt{evi'}, \texttt{inn})$	no: The argument is not coherent.
$(\texttt{evi} \wedge \texttt{evi'}, \texttt{gui})$	yes: $\Pr(\text{Case 4} \mid \texttt{evi} \wedge \texttt{evi'}) = 1$

which there is a witness, but no DNA match), that can be reflected in $\Pr$. Whichever representation $\Pr$ as in Corollary 2 is chosen, the probability calculus is followed. Hence Bayes' theorem and its odds version using a likelihood ratio hold.

5 Example: Alfred Hitchcock's 'To Catch A Thief'

As an example of the development of evidential reasoning in which gradually information is collected, we discuss the crime investigation story that is the backbone of Alfred Hitchcock's 'To Catch A Thief', otherwise—what Hitchcock himself referred to as—a lightweight story about a French Riviera love affair, starring Grace Kelly and Cary Grant. In the film, Grant plays a former robber Robie, called 'The Cat' because of his spectacular robberies, involving the climbing of high buildings. At the beginning of the film, new 'The Cat'-like thefts have occurred. Because of this resemblance with Robie's style (the first evidence considered, denoted in what follows as $\texttt{res}$), the police consider the hypothesis that Robie is again the thief ($\texttt{rob}$), and also that he is not ($\neg\texttt{rob}$). Figure 8 provides a graphical representation of the investigation. The first row shows the situation after the first evidence $\texttt{res}$, mentioned on the left side of the figure, with the two hypothical conclusions $\texttt{rob}$ and $\neg\texttt{rob}$ represented as rectangles. The size of a rectangle's area suggests the strength of the argument from the accumulated evidence to the hypothesis. Here the arguments from $\texttt{res}$ to $\texttt{rob}$ and $\neg\texttt{rob}$ are of comparable strength.

When the police confront Robie with the new thefts, he escapes with the goal to catch the real thief. By this second evidence ($\texttt{esc}$), the hypothesis $\texttt{rob}$ becomes more strongly supported than its opposite $\neg\texttt{rob}$. In the figure, the

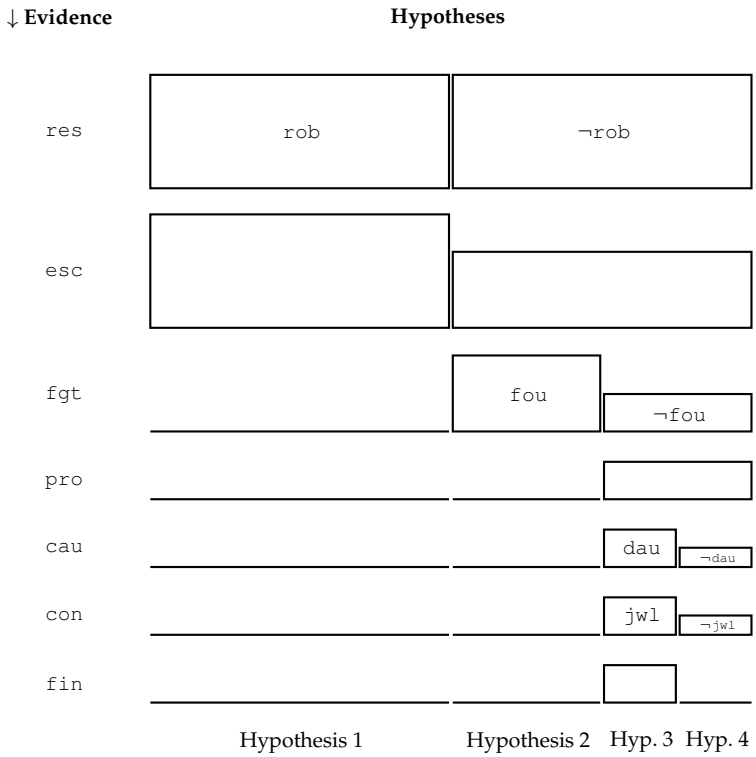

Fig. 8 Example: Hitchcock's 'To Catch A Thief'

second row indicates the situation after the two pieces of evidence are available. As indicated by the rectangles of differently sized areas, the argument from the accumulated evidence res∧esc to rob is stronger than that from the same premises to ¬rob. Rectangles in a column in the figure represent corresponding hypotheses. Sentences shown in a corresponding hypothesis in a higher row are not repeated. So on the second row, when the evidence res and esc are taken into account, the rectangles correspond to rob (on the left) and ¬rob (on the right).

Robie sets a trap for the real thief, resulting in a night-time fight on the roof with Foussard who falls and dies (fgt). The police consider this strong evidence for the hypothesis that Foussard is the thief (fou), but not conclusive so also the opposite hypothesis is considered coherent (¬fou). In the figure (third row marked fgt) the hypothesis ¬rob is split into two hypotheses: one rectangle representing ¬rob∧fou, the other ¬rob∧¬fou, both in conjunction with the evidence available at this stage of the investigation (res∧esc ∧fgt). With the accumulated evidence res∧esc ∧fgt as premises, the hypothesis ¬rob∧fou is more strongly supported than the hypothesis ¬rob∧¬fou. The police no longer believe that Robie is the thief. This is indicated by the line on the left of the third row in the figure. The

premises res∧esc ∧fgt do not provide support for the hypothesis rob; or, in the terminology of this paper: the argument from premises res∧esc ∧fgt to conclusion rob is not coherent.

Robie points out that Foussard cannot be the new incarnation of 'The Cat', as he had a prosthetic wooden leg (pro). In other words, the argument from res∧esc ∧fgt∧pro to ¬rob∧fou is not coherent. (Cf. the second line in the fourth row of the figure, corresponding to the hypothesis that Foussard is the thief.)

Later in the film, Foussard's daughter is caught in the act (cau), providing very strong support for the hypothesis that the daughter is the new cat (dau). The argument from res∧esc ∧fgt∧pro∧cau to dau is stronger than to ¬dau.

In her confession (con), Foussard's daughter explains where the jewelry stolen earlier can be found, adding some specific information to the circumstances of her crimes (jwl). The argument from res∧esc ∧fgt∧pro∧cau ∧con to dau ∧ jwl is stronger than to ¬dau ∧ ¬jwl.

The police find the jewelry at the indicated place (fin) and there is no remaining doubt about the hypothesis that Foussard's daughter is the thief. The argument from res∧esc ∧fgt∧pro∧cau∧con ∧fin to ¬dau∧ ¬jwl is incoherent, as indicated by the line on the right of the bottom row of the figure. In the only remaining hypothesis, Foussard's daughter is the thief, and not Robie, and not Foussard. In other words, the argument from res∧esc ∧fgt∧pro∧cau∧con ∧jwl to ¬rob∧¬fou∧dau is conclusive.

During the investigation, gradually a case model has been developed representing the arguments discussed in the example. We distinguish 7 cases, as follows:

Case 1: rob
∧ res∧esc
Case 2: ¬rob∧fou
∧ res∧esc∧fgt
Case 3: ¬rob∧¬fou∧dau∧jwl
∧ res∧esc∧fgt∧pro∧cau∧con∧fin
Case 4: ¬rob∧¬fou∧¬dau∧¬jwl
∧ res∧esc∧fgt∧pro∧cau∧con
Case 5: ¬rob
∧res∧¬esc
Case 6: ¬rob∧¬fou
∧res∧esc∧¬fgt
Case 7: ¬rob∧¬fou∧¬dau
∧res∧esc∧fgt∧pro∧¬cau

Cases 1 to 4 are found as follows. First the properties of the four main hypotheses are accumulated from the columns in Figure 8:

Hypothesis 1: rob
Hypothesis 2: ¬rob∧fou

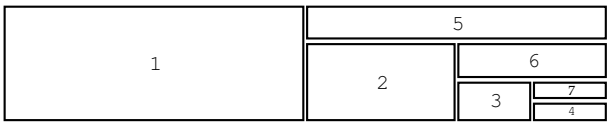

Fig. 9 Case model for the example

Hypothesis 3: ¬rob∧¬fou∧dau∧jwl
Hypothesis 4: ¬rob∧¬fou∧¬dau∧¬jwl

Then these are conjoined with the maximally specific accumulated evidence that provide a coherent argument for them:

Evidence coherent with hypothesis 1: res∧esc
Evidence coherent with hypothesis 2: res∧esc∧fgt
Evidence coherent with hypothesis 3: res∧esc∧fgt∧pro∧cau∧con∧fin
Evidence coherent with hypothesis 4: res∧ esc∧fgt∧pro∧cau∧con

The cases 5 to 7 complete the case model. Case 5 is the hypothetical case that Robie is not the thief, that there is resemblance, and the Robie does not escape. In Case 6, Robie and Foussard are not the thieves, and there is no fight. In Case 7, Robie, Foussard and his daughter are not the thieves, and she is not caught in the act. Note that the cases are consistent and mutually exclusive.

Figure 9 shows the 7 cases of the model. The sizes of the rectangles represent the preferences. The preference relation has the following equivalence classes, ordered from least preferred to most preferred:

1. Cases 4 and 7;
2. Case 3;
3. Cases 2 and 6;
4. Cases 1 and 5.

Note that the rectangles in Figure 8 can be constructed as combinations of the rectangles in Figure 9.

The discussion of the arguments, their coherence, conclusiveness and validity presented semi-formally above fits this case model. For instance, the argument from the evidential premises res∧esc to the hypothesis rob is presumptively valid in this case model since Case 1 is the only case implying the case made by the argument. It is not conclusive since also the argument from these same premises to ¬rob is coherent. The latter argument is not presumptively valid since all ¬rob-cases implying the premises (Cases 2 to 7) have lower preference than Case 1. The argument from res∧esc∧fgt to rob is incoherent as there is no case in which the premises and the conclusion follow. Also arguments that do not start from evidential premises can be evaluated. For instance, the argument from the premise (not itself evidence) dau to jwl is conclusive since in the only case implying the premises (Case 3) the conclusion follows. Finally we find the conclusive argument from premises res∧esc∧fgt∧pro∧cau∧con∧jwl to conclusion ¬rob∧¬fou∧dau∧jwl (only Case 3 implies the premises), hence also to dau.

6 Concluding remarks

In this paper, we have discussed correct reasoning with evidence using three analytic tools: arguments, scenarios and probabilities. We proposed a formalism in which the presumptive validity of arguments is defined in terms of case models, and studied properties (Section 3). In particular, we showed that the qualitative definitions of case models and presumptive validity have a quantitative representation in terms of probability functions. We discussed key concepts in the argumentative, scenario and probabilistic analysis of reasoning with evidence in terms of the formalism (Section 4). An example of the gradual development of evidential reasoning was provided in Section 5.

This work builds on a growing literature aiming to formally connect the three analytic tools of arguments, scenarios and probabilities. In a discussion of the anchored narratives theory by Crombag, Wagenaar and Van Koppen (1993), it was shown how argumentative notions were relevant in their scenario analyses (Verheij, 2000). Bex has provided a hybrid model connecting arguments and scenarios (Bex, 2011; Bex et al., 2010), and has worked on the further integration of the two tools (Bex, 2015; Bex and Verheij, 2013). Connections between arguments and probabilities have been studied by Hepler, Dawid and Leucari (2007) combining object-oriented modeling and Bayesian networks. Fenton, Neil and Lagnado continued this work by developing representational idioms for the modeling of evidential reasoning in Bayesian networks (Fenton et al., 2013). Inspired by this research, Vlek developed scenario idioms for the design of evidential Bayesian networks containing scenarios (Vlek et al., 2014), and Timmer showed how argumentative information can be extracted from a Bayesian network (Timmer et al., 2015b). Keppens and Schafer (Keppens and Schafer, 2006) studied the knowledge-based generation of hypothetical scenarios for reasoning with evidence, later developed further in a decision support system (Shen et al., 2006).

This paper continues from an integrated perspective on arguments, scenarios and probabilities (Verheij, 2014b). In the present paper, that integrated perspective is formally developed (building on ideas in Verheij, 2014a) using case models and discussing key concepts used in argumentative, scenario and probabilistic analyses. Interestingly, our case models and their preferences are qualitative in nature, while the preferences correspond exactly to those that can be numerically and probabilistically realized. As such, the present formal tools combine a non-numeric and numeric perspective (cf. the paper 'To Catch A Thief With and Without Numbers'; Verheij, 2014b). The mathematics of the formalism is studied further in (Verheij, 2016a) and has been applied to value-guided decision making in (Verheij, 2016c).

The present work does not require modeling evidential reasoning in terms of full probability functions, as is the case in Bayesian network approaches. In this way, the well-known problem of needing to specify more numbers than are reasonably available is addressed. In fact, we have shown an approach in which the specific numbers of a quantitative representation can be abstracted to a qualitative representation. Also whereas the causal interpre-

tation of Bayesian networks is risky (Dawid, 2010), our case models come with formal definitions of arguments, their coherence, conclusiveness and presumptive validity.

From a knowledge representation perspective, one relevant question is what happens in more complex examples than the ones used here. Indeed, more realistic examples can quickly increase in complexity and may lead to more cases than can be handled. This question has not been addressed in this paper. A helpful next step could be to perform a case study of a real example, but also the formal investigation of the growth of complexity can prove fruitful.

Another knowledge representation issue is where the case models come from. In the formal proposal in this paper, the evaluation of arguments and scenarios happens against the background of a given case model. So such evaluation requires that a case model is available. No systematic approach for the development of case models is discussed in this paper. For the Bayesian network modeling of scenarios, Vlek et al. (2014) provides such a method, and Timmer et al. (2015a) discusses the inclusion of argumentation schemes and their critical questions in a Bayesian network model. These works continue from the use of building blocks and idioms for building a Bayesian network model of the evidence in a criminal case, pioneered by Hepler et al. (2007) and Fenton et al. (2013). Perhaps ideas from these systematic modeling approaches can be adapted to the present setting.

By the present and related studies, we see a gradual clarification of how arguments, scenarios and probabilities all have their specific useful place in the analysis of evidential reasoning. By explicating formal bridges between qualitative and quantitative analytic styles, we have provided an explanation why some prefer to rationally analyze proof numerically, and others non-numerically. As a result, it seems ever less natural to choose between the three kinds of tools, and ever more so to use each of them when practically applicable.

Acknowledgements This research has been performed in the context of the project 'Designing and Understanding Forensic Bayesian Networks with Arguments and Scenarios', funded in the NWO Forensic Science program (`http://www.ai.rug.nl/~verheij/nwofs/`). A version of the material was presented at the *ECAI 2016 Workshop on Artificial Intelligence for Justice*, August 30, 2016, The Hague (Verheij, 2016b). The author would like to thank the Isaac Newton Institute for Mathematical Sciences (University of Cambridge) for its hospitality during the programme 'Probability and Statistics in Forensic Science' which was supported by EPSRC Grant Number EP/K032208/1.

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