Bever Gang - Axioms

(Multi-Agent Systems Project)

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Axiom system

For every action, knowledge of the agents is updated using predefined axioms. Because every action yields different update rules, the different axioms we've used in our project implementation are divided into different categories corresponding to the different actions. Every action can be divided into two sections: *myAction* and *othersAction*. myAction corresponds to the action performed by the agent itself, whereas othersAction corresponds to the action performed by some other agent.

Terminology

Throughout this overview, we denote agent i as the agent itself, and denote agent o as some other agent. By using another letter from the alphabet we mean that it can be any agent.

$card_{(x,y)}$	The card of agent x on position y
$card_{open}$	The current card on top of the open pile
$card_{(x,\forall)}$	All the cards of agent x
$card_{(x,y\neq z)}$	Any card y of agent x, except the one on position z
$\bar{x}$	The average card value in the deck (usually 5.0)
$V(card_{pos})$	The value of the card on position *pos*
$\bar{V}(card_{pos})$	The average value of the card on position *pos**
$K_x\varphi$	Agent x knows φ
$K_x(card_{pos} = \mathbf{t})$	Agent x knows the value of the card on position *pos*
$K_x(card_{pos} = \mathbf{f})$	Agent x does not know the value of the card on position *pos*
$card_{pos} \neq S$	The card on position *pos* is not a special card (not a 'Peek', 'Swap' or 'GetTwo')

*To clarify: Suppose that only cards 0-9 are in play and of every card there is equally many. Then, when an agent knows that the value of a card must be lower than or equal to 6, the average value of that card would be 3. When an agent knows that the exact value of a card, the average would be that exact value.

1 Draw a card from the open pile

Drawing a card from the open pile does not involve updating knowledge. Both the agent itself and the others do not gain any information by it, except of course that they now know the value of the card in the agent's hand (this is used by putting a card on a position, Section 7).

2 Draw a card from the closed pile

Drawing a card from the closed pile does involve updating knowledge, but only in the situations where the card on the open pile was an interesting card.

2.1 myAction

In this case, the agent itself drew a card from the closed pile. Here, only 2nd and 3rd depth knowledge is updated:

$$((card_{open} \neq S) \wedge (V(card_{open}) < \bar{x}) \wedge K_i K_x K_i(card_{(i,y)} = \mathbf{t})) \rightarrow K_i K_x(V(card_{(i,y)}) \leq V(card_{open})) \tag{2.1}$$

$$((card_{open} \neq S) \wedge (V(card_{open}) < \bar{x}) \wedge K_i K_x K_i(card_{(i,y)} = \mathbf{t})) \rightarrow K_i K_x K_i(V(card_{(i,y)}) \leq V(card_{open})) \tag{2.2}$$

If the open card is not a special card and the value of the open card is lower than average, and I know that another agent knows that I know the value of one of my cards, then I know that the other agent knows that the value of my card must be lower than or equal to the value of the open card, otherwise, I would have drawn the open card (Axiom 2.1). Likewise for depth 3, now I know that the other agent knows that I know that the value of my card is lower than or equal to the value of the open card (Ax. 2.2).

2.2 othersAction

In this case, some other agent drew a card from the closed pile. Here, the agent that performed the action is indicated by o (other).

$$((card_{open} \neq S) \wedge (V(card_{open}) < \bar{x}) \wedge K_i K_o(card_{(o,x)} = \mathbf{t})) \rightarrow K_i(V(card_{(o,x)}) \leq V(card_{open})) \tag{2.3}$$

$$((card_{open} \neq S) \wedge (V(card_{open}) < \bar{x}) \wedge K_i K_o(card_{(o,x)} = \mathbf{t})) \rightarrow K_i K_o(V(card_{(o,x)}) \leq V(card_{open})) \tag{2.4}$$

If the open card is not a special card and the value of the open card is lower than average, and I know that agent o knows one of his cards, then I know that the value of that card must be lower than or equal to the value of the open card (Ax. 2.3). Otherwise, the other agent would have picked the open card. Now, I also know that the other agent knows that the value of his card is lower than or equal to the value of the open card (Ax. 2.4).

$$((card_{open} \neq S) \wedge (V(card_{open}) < \bar{x}) \wedge K_i K_y K_o(card_{(o,x)} = \mathbf{t})) \rightarrow K_i K_y(V(card_{(o,x)}) \leq V(card_{open})) \tag{2.5}$$

$$((card_{open} \neq S) \wedge (V(card_{open}) < \bar{x}) \wedge K_i K_y K_o(card_{(o,x)} = \mathbf{t})) \rightarrow K_i K_y K_o(V(card_{(o,x)}) \leq V(card_{open})) \tag{2.6}$$

If the open card is not a special card and the value of the open card is lower than average, and I know that an agent y knows that agent o knows one of his card, then I now know that agent y knows that the value of that card is lower than or equal to the value of the open card (Ax. 2.5). Also, I now know that an agent y knows that agent o knows that the value of his card is lower than or equal to the value of the open card (Ax. 2.6).

3 Discard

The update rules for discarding a card are similar to that of drawing a card from the closed pile. The only difference is that the discarded card (which will become the top most card of the open pile) does not have to be lower than average for the update rules to apply. Let's for example consider the update rule 2.1, but then modified to fit this action:

$$(card_{open} \neq S) \wedge K_i K_x K_i(card_{(i,y)} = \mathbf{t})) \rightarrow K_i K_x (V(card_{(i,y)}) \leq V(card_{open})) \tag{3.1}$$

Here, I discard the card. Because I discarded the card, the discard card is now the open card. Now, when I know that another agent knows that I know one of my cards, then I know that he knows that the value of that card is lower than or equal to the value of the open card (Ax. 3.1). The other axioms are obtained similarly.

4 Peek

4.1 myAction

Here, I peeked one of my cards. The card that I peeked is indicated by the position (i, b) and the value of the peeked card is indicated by v. The update rules are then as follows:

$$K_i(V(card_{(i,b)}) = v) \tag{4.1}$$

$$K_i K_x K_i(card_{(i,b)} = \mathbf{t}) \tag{4.2}$$

Now I know the exact value of my peeked card (Ax. 4.1). Also, I know that another agent knows I know the value of my peeked card (Ax. 4.2).

4.2 othersAction

Here, some other agent o peeked one of his cards on position (o, b).

$$K_i K_o(card_{(o,b)} = \mathbf{t}) \tag{4.3}$$

$$K_i K_x K_o(card_{(o,b)} = \mathbf{t}) \tag{4.4}$$

Now I know that the agent o knows the value of his card (Ax. 4.3). Also, I know that another agent knows that agent o knows the value of his card (Ax. 4.4).

5 GetTwo

'GetTwo' has no extra axioms in it. It combines axioms from different actions, depending on the actions the agent performs. Let's look at one example:

An agent plays the 'GetTwo' card and has to draw a card from the closed pile, here no knowledge is updated because the agent was forced to draw from the closed pile. Now, the agent must decide if he wants to keep the first card or draw a second. Discarding the first card and drawing a second results in the same update rules as 'draw from closed'. The update rules for playing a card are also similar to just playing a card.

6 Swap

Swapping cards requires a lot of difficult axioms, because the reasons to swap can differ in many ways. The first axioms, however, are quite simple and not shown here: for both *myAction* and *othersAction*, swap all the knowledge of the first card (pos) with the knowledge of the second card (pos').

6.1 myAction

Here, I swapped a card of my own on position (i, a) with a card of another agent on position (o, b). The update rules shown here are after the action was performed, the cards are already swapped. This means that the card that first lay on position (i, a) now lies on position (o, b) and vice versa.

$$(K_iK_xK_i((card_{(i,a)} = \mathbf{t}) \wedge (card_{(o,b)} = \mathbf{t})) \wedge K_iK_x(V(card_{(i,a)}) = v)) \rightarrow K_iK_x(V(card_{(o,b)}) \geq v) \quad (6.1)$$
$$(K_iK_xK_i((card_{(i,a)} = \mathbf{t}) \wedge (card_{(o,b)} = \mathbf{t})) \wedge K_iK_x(V(card_{(i,a)}) = v)) \rightarrow K_iK_xK_i(V(card_{(o,b)}) \geq v) \quad (6.2)$$

If I know that an agent x knows that I know the value of both cards, and I know that he knows the value of the card on position (i, a) is v (which first lay on position (o, b)), then I now know that he knows that the value of the card on position (o, b) is equal to or higher than v (Ax. 6.1). Otherwise, I wouldn't have swapped those cards. Also, I now know that agent x knows that I know that the value of the card on position (o, b) is equal to or higher than v (Ax. 6.2).

$$(K_iK_xK_i((card_{(i,\forall\neq a)} = \mathbf{true}) \wedge (card_{(o,b)} = \mathbf{t})) \wedge K_iK_x(card_{(i,y\neq a)} = \mathbf{t})) \rightarrow \\ K_iK_x(V(card_{(o,b)}) \geq K_iK_x(V(card_{(i,y)}))) \quad (6.3)$$
$$(K_iK_xK_i((card_{(i,\forall\neq a)} = \mathbf{true}) \wedge (card_{(o,b)} = \mathbf{t})) \wedge K_iK_x(card_{(i,y\neq a)} = \mathbf{t})) \rightarrow \\ K_iK_xK_i(V(card_{(o,b)}) \geq K_iK_x(V(card_{(i,y)}))) \quad (6.4)$$

If I know that an agent x knows that I knew all of my cards before swapping, and I know that agent x knows my card on position (i, y), where y is not a (so not the card just swapped), then I know that agent x knows that the value of the card on position (o, b) is higher than or equal to the value I know that he knows of that card on position (i, y) (Ax. 6.3). Otherwise, I wouldn't have swapped the cards on positions (i, a) and (o, b). Also, I now know that agent x knows that I know that the value of the card on position (o, b) is higher than or equal to the value I know that he knows of the card on position (i, y) (Ax. 6.4).

$$K_iK_xK_i((card_{(i,\forall\neq a)} = \mathbf{true}) \wedge (card_{(o,b)} = \mathbf{t})) \rightarrow K_iK_x(V(card_{(i,\forall\neq a)}) \leq K_iK_x(V(card_{(o,b)}))) \quad (6.5)$$
$$K_iK_xK_i((card_{(i,\forall\neq a)} = \mathbf{true}) \wedge (card_{(o,b)} = \mathbf{t})) \rightarrow K_iK_xK_i(V(card_{(i,\forall\neq a)}) \leq K_iK_x(V(card_{(o,b)}))) \quad (6.6)$$

If I know that an agent x knows that I knew all of my cards before swapping, then I know that agent x knows that the value of all my cards, except the one I just received with swapping, is lower than or equal to the value I know that he knows of the card on position (o, b) (Ax. 6.5). Otherwise, I wouldn't have swapped those cards. Also, I now know that agent x knows that I know that the value of all my cards, except the one on position (i, a), is lower than or equal to the value I know that agent x knows of the card on position (o, b) (Ax. 6.6).

$$(K_iK_xK_i((card_{(i,a)} = \mathbf{t}) \wedge (card_{(o,b)} = \mathbf{t})) \wedge K_iK_x(V(card_{(o,b)}) = v)) \rightarrow K_iK_x(V(card_{i,a}) \leq v) \quad (6.7)$$
$$(K_iK_xK_i((card_{(i,a)} = \mathbf{t}) \wedge (card_{(o,b)} = \mathbf{t})) \wedge K_iK_x(V(card_{(o,b)}) = v)) \rightarrow K_iK_xK_i(V(card_{i,a}) \leq v) \quad (6.8)$$

If I know that agent x knows that I know both cards, and I know that agent x knows the value of the card on position (o, b) is v, then I know that agent x knows that the value of the card on position (i, a) is lower than or equal to v (Ax. 6.7). Otherwise, I wouldn't have swapped those cards. Also, I now know that agent x knows that I know that the value of the card on position (i, a) is lower than or equal to v (Ax. 6.8).

6.2 othersAction

Here, some other agent o swapped two cards, his card on position (o, b) with another card on position (x, a). Here we use x, meaning that it does not have to be one of my cards (i) that was being swapped. Also, the action is already performed, meaning that the cards have already been swapped.

$$(K_iK_o((card_{(o,b)} = \mathbf{t}) \wedge (card_{(x,a)} = \mathbf{t})) \wedge K_i(V(card_{(o,b)}) = v)) \rightarrow K_i(V(card_{(x,a)}) \geq v) \tag{6.9}$$

$$(K_iK_o((card_{(o,b)} = \mathbf{t}) \wedge (card_{(x,a)} = \mathbf{t})) \wedge K_i(V(card_{(o,b)}) = v)) \rightarrow K_iK_o(V(card_{(x,a)}) \geq v) \tag{6.10}$$

If I know that the other agent o knows both cards, and I know the value of the card on position (o, b) is v, then I know that the value of the card on position (x, a) is higher then or equal to v (Ax. 6.9). Otherwise, the agent wouldn't have swapped those cards. Also, I now know that the other agent knows that the value on position (x, a) is higher then or equal to v (Ax. 6.10).

$$(K_iK_o((card_{(o,\forall\neq b)} = \mathbf{t}) \wedge (card_{(x,a)} = \mathbf{t})) \wedge K_i(card_{(o,y\neq b)} = \mathbf{t})) \rightarrow K_i(V(card_{(x,a)}) \geq K_i(V(card_{(o,y)}))) \tag{6.11}$$

$$(K_iK_o((card_{(o,\forall\neq b)} = \mathbf{t}) \wedge (card_{(x,a)} = \mathbf{t})) \wedge K_i(card_{(o,y\neq b)} = \mathbf{t})) \rightarrow K_iK_o(V(card_{(x,a)}) \geq K_i(V(card_{(o,y)}))) \tag{6.12}$$

If I know that the other agent o knew all of his cards before swapping, and I know a card y of agent o except the card that he just swapped, then I know that the value of the card on position (x, a) is higher than or equal to the value I know of the card on position (o, y) (Ax. 6.11). Also, I now know that the other agent o knows that the value of the card on position (x, a) is higher than or equal to the value I know of the card on position (o, y) (Ax. 6.12).

$$K_iK_o((card_{(o,\forall\neq b)} = \mathbf{t}) \wedge (card_{(x,a)} = \mathbf{t})) \rightarrow K_i(V(card_{(o,\forall\neq b)}) \leq K_i(V(card_{(x,a)}))) \tag{6.13}$$

$$K_iK_o((card_{(o,\forall\neq b)} = \mathbf{t}) \wedge (card_{(x,a)} = \mathbf{t})) \rightarrow K_iK_o(V(card_{(o,\forall\neq b)}) \leq K_i(V(card_{(x,a)}))) \tag{6.14}$$

If I know that the other agent o knew all of his cards before swapping, then I know that the value of all of agent o's cards, except the one that he swapped, is lower than or equal to the value I know of the card on position (x, a) (Ax. 6.13). Also, I now know that the other agent o knows that the value of all of his cards, except the one that he swapped, is lower than or equal to the value I know of the card on position (x, a) (Ax. 6.14).

$$(K_iK_o((card_{(o,b)} = \mathbf{t}) \wedge (card_{(x,a)} = \mathbf{t})) \wedge K_i(V(card_{(x,a)}) = v)) \rightarrow K_i(V(card_{(o,b)}) \leq v) \tag{6.15}$$

$$(K_iK_o((card_{(o,b)} = \mathbf{t}) \wedge (card_{(x,a)} = \mathbf{t})) \wedge K_i(V(card_{(x,a)}) = v)) \rightarrow K_iK_o(V(card_{(o,b)}) \leq v) \tag{6.16}$$

If I know that the other agent o knows the cards that are being swapped, and I know the value of the card on position (x, a) is v, then I know that the value of the card on position (o, b) is lower than or equal to v (Ax. 6.15). Otherwise, the other agent wouldn't have swapped those cards. Similarly, I now know that the other agent knows that the value of the card on position (o, b) is lower than or equal to v (Ax. 6.16).

$$(K_iK_yK_o((card_{(o,b)} = \mathbf{t}) \wedge (card_{(a,x)} = \mathbf{t})) \wedge K_iK_y(card_{(o,b)} = \mathbf{t}) \wedge K_i(V(card_{(o,b)}) = v)) \rightarrow K_iK_y(V(card_{(a,x)}) \geq v) \tag{6.17}$$

$$(K_iK_yK_o((card_{(o,b)} = \mathbf{t}) \wedge (card_{(a,x)} = \mathbf{t})) \wedge K_iK_y(card_{(o,b)} = \mathbf{t}) \wedge K_i(V(card_{(o,b)}) = v)) \rightarrow K_iK_yK_o(V(card_{(a,x)}) \geq v) \tag{6.18}$$

If I know that an agent y knows that agent o knows the value of both cards that are being swapped, and I know that agent y knows the value of the card on position (o, b), and I know that the value of the card on position (o, b) is v, then I know that agent y knows that the value of the card on position (a, x) is higher than or equal to v (Ax. 6.17). Likewise, I now know that agent y knows that agent o knows that the value of the card on position (a, x) is higher than or equal to v (Ax. 6.18).

$$(K_iK_yK_o((card_{(o,b)} = \mathbf{t}) \wedge (card_{(a,x)} = \mathbf{t})) \wedge K_iK_y(card_{(a,x)} = \mathbf{t}) \wedge K_i(V(card_{(a,x)}) = v)) \rightarrow K_iK_y(V(card_{(o,b)}) \leq v) \tag{6.19}$$

$$(K_iK_yK_o((card_{(o,b)} = \mathbf{t}) \wedge (card_{(a,x)} = \mathbf{t})) \wedge K_iK_y(card_{(a,x)} = \mathbf{t}) \wedge K_i(V(card_{(a,x)}) = v)) \rightarrow K_iK_yK_o(V(card_{(o,b)}) \leq v) \tag{6.20}$$

If I know that an agent y knows that agent o knows the value of both cards that are being swapped, and I know that agent y knows the value of the card on position (a, x), and I know that the value of the card on position (a, x) is v, then I know that agent y knows that the value of the card on position (o, b) is lower than or equal to v (Ax. 6.19). Likewise, I now know that agent y knows that agent o knows that the value of the card on position (o, b) is lower than or equal to v (Ax. 6.20).

7 Put a card on a position

As with swap, the action 'put' has a lot of difficult update rules. The update rules for 'put' differs in the fact that an agent can put a card from the open pile or closed pile. Both situations give different knowledge about the cards of the agent. Therefore, next to dividing the section into *myAction* and *othersAction*, we've divided this section into three more components: one for putting a card from the open pile; one for putting a card from the closed pile; and one for update rules that apply in both cases.

7.1 myAction

7.1.1 Put from open pile

Here, the agent itself put a card from the open pile. The downside of this is that every agent knows beforehand what the value of that card was. Here, we denote the position of the put card as (i, a) and the value of that card as v. Remember that the action is already performed, therefore, the discarded card is now on top of the open pile. Also, we assume that every agent knew the previous open pile card.

$$K_i(V(card_{(i,a)}) = v) \tag{7.1}$$
$$K_iK_x(V(card_{(i,a)}) = v) \tag{7.2}$$
$$K_iK_xK_y(V(card_{(i,a)}) = v) \tag{7.3}$$

I now know that the value of the card on position (i, a) is v (Ax. 7.1). Also, I know that another agent x knows that (Ax. 7.2), and I know that another agent x knows that another agent y knows that (Ax. 7.3).

7.1.2 Put from closed pile

Here, the agent itself put a card from the closed pile. Again we denote the position of the put card as (i, a) and the value of that card as v. In addition, we denote the discarded card from a position (x, y) as (x', y'). We use this notation to be able to refer to what an agent knew about that card, before discarding it. To clarify, $K_i(card_{(x',y')} = \mathbf{t})$ is read as: agent i knew the value of the card which was previously on position (x, y). Recall that $\bar{V}(\varphi)$ stands for the average value of φ.

$$K_i(V(card_{(i,a)}) = v) \tag{7.4}$$
$$K_iK_xK_i(card_{(i,a)} = \mathbf{t}) \tag{7.5}$$

Of course, I now know the value of my new card on position (i, a) (Ax. 7.4). Also, I know that another agent x knows that I know the value of the card on position (i, a) (Ax. 7.5).

$$K_iK_xK_i(card_{(i',a')} = \mathbf{t}) \to K_iK_x(V(card_{(i,a)}) \leq V(card_{open})) \tag{7.6}$$
$$K_iK_xK_i(card_{(i',a')} = \mathbf{t}) \to K_iK_xK_i(V(card_{(i,a)}) \leq V(card_{open})) \tag{7.7}$$

If I know that another agent x knows that I knew the card previously on position (i, a), then I know that agent x knows that the value of the current card on position (i, a) is lower than or equal to the value of the open pile card (Ax. 7.6). Similarly, I know that agent x knows that I know that the value of the current card on position (i, a) is lower than or equal to the value of the open pile card (Ax. 7.7).

$$K_iK_xK_i(card_{(i',a')} = \mathbf{f}) \rightarrow K_iK_x(V(card_{(i,a)}) \leq K_iK_xK_i(\bar{V}(card_{(i',a')}))) \quad (7.8)$$

$$K_iK_xK_i(card_{(i',a')} = \mathbf{f}) \rightarrow K_iK_xK_i(V(card_{(i,a)}) \leq K_iK_xK_i(\bar{V}(card_{(i',a')}))) \quad (7.9)$$

In contrast, when I know that another agent x knows that I did not knew the card previously on position (i,a), then I know that agent x knows that the value of the current card on position (i,a) is lower than or equal to the average value I know agent x knows I knew of the card previously on position (i,a) (Ax. 7.8). Note that, for example, when an agent x does not know the value of a card, he can know that the value of the card is lower than some value. Also, I now know that agent x knows that I know that the value of the current card on position (i,a) is lower than or equal to the average value I know agent x knows I knew of the card previously on position (i,a) (Ax. 7.9).

7.1.3 Update rules for both cases

$$K_iK_xK_i((card_{(i',a')} = \mathbf{t}) \wedge (card_{(i,y \neq a)} = \mathbf{t})) \rightarrow K_iK_x(V(card_{(i,y)}) \leq V(card_{open})) \quad (7.10)$$

$$K_iK_xK_i((card_{(i',a')} = \mathbf{t}) \wedge (card_{(i,y \neq a)} = \mathbf{t})) \rightarrow K_iK_xK_i(V(card_{(i,y)}) \leq V(card_{open})) \quad (7.11)$$

If I know that agent x knows that I knew the card previously on position (i,a), and I know that agent x knows that I know my card on position (i,y), where y is not a (so not the card that I just put), then I know that agent x knows that the value of my card (i,y) is lower than or equal to the value of the open pile card (Ax. 7.10). Also, I now know that agent x knows that I know that the value of my card on position (i,y) is lower than or equal to the value of the open pile card (Ax. 7.11).

$$K_iK_xK_i((card_{(i',a')} = \mathbf{f}) \wedge (card_{(i,y \neq a)} = \mathbf{t})) \rightarrow K_iK_x(V(card_{(i,y)}) \leq K_iK_xK_i(\bar{V}(card_{(i',a')}))) \quad (7.12)$$

$$K_iK_xK_i((card_{(i',a')} = \mathbf{f}) \wedge (card_{(i,y \neq a)} = \mathbf{t})) \rightarrow K_iK_xK_i(V(card_{(i,y)}) \leq K_iK_xK_i(\bar{V}(card_{(i',a')}))) \quad (7.13)$$

If I know that agent x knows that I did not knew the card previously on position (i,a), and I know that agent x knows that I know my card on position (i,y), where y is not a (so not the card that I just put), then I know that agent x knows that the value of the card on position (i,y) is lower than or equal to the average value I know agent x knows I knew of the card previously on position (i,a) (Ax. 7.12). Similarly, I know that agent x knows that I know that the value of the card on position (i,y) is lower than or equal to the average value I know agent x knows I knew of the card previously on position (i,a) (Ax. 7.13).

7.2 othersAction

7.2.1 Put from open pile

Here, some other agent o put a card from the open pile. We denote the position of the put card as (o,b) and the value of that card as v.

$$K_i(V(card_{(o,b)}) = v) \quad (7.14)$$

$$K_iK_x(V(card_{(o,b)}) = v) \quad (7.15)$$

$$K_iK_xK_y(V(card_{(o,b)}) = v) \quad (7.16)$$

I now know that the value of the card on position (o,b) is v (Ax. 7.14). Also, I know that another agent x knows that (Ax. 7.15), and I know that another agent x knows that another agent y knows that (Ax. 7.16).

7.2.2 Put from closed pile

Here, some other agent o put a card from the closed pile. We denote the position of the put card as (o,b) and the value of that card as v. Again, we use (x',y') to refer to the card previously (before the action) on position (x,y).

$$K_iK_o(card_{(o,b)} = \mathbf{t}) \tag{7.17}$$

Now I know that agent o knows the card on position (o, b) (Ax. 7.17).

$$K_iK_o(card_{(o',b')} = \mathbf{t}) \rightarrow K_i(V(card_{(o,b)}) \leq V(card_{open})) \tag{7.18}$$
$$K_iK_o(card_{(o',b')} = \mathbf{t}) \rightarrow K_iK_o(V(card_{(o,b)}) \leq V(card_{open})) \tag{7.19}$$

If I know that agent o knew the card previously on position (o, b), then I know that the value of the card on position (o, b) is lower than or equal to the value of the open pile card (which is the discarded card) (Ax. 7.18). Also, I know that agent o knows that his card on position (o, b) is lower than or equal to the value of the open pile card (Ax. 7.19).

$$K_iK_xK_o(card_{(o',b')} = \mathbf{t}) \rightarrow K_iK_x(V(card_{(o,b)}) \leq V(card_{open})) \tag{7.20}$$
$$K_iK_xK_o(card_{(o',b')} = \mathbf{t}) \rightarrow K_iK_xK_o(V(card_{(o,b)}) \leq V(card_{open})) \tag{7.21}$$

Similar to the previous rules, if I know another agent x knows that o knew the card previously on position (o, b), then I know that agent x knows that the value of the card on position (o, b) is lower than or equal to the value of the open pile card (Ax. 7.20). And, I now know that agent x knows that agent o knows that his card on position (o, b) is lower than or equal to the value of the open pile card (Ax. 7.21).

$$K_iK_o(card_{(o',b')} = \mathbf{f}) \rightarrow K_i(V(card_{(o,b)}) \leq K_iK_o\bar{V}(card_{(o',b')})) \tag{7.22}$$
$$K_iK_o(card_{(o',b')} = \mathbf{f}) \rightarrow K_iK_o(V(card_{(o,b)}) \leq K_iK_o\bar{V}(card_{(o',b')})) \tag{7.23}$$

If I know that agent o did not knew the card previously on position (o, b), then I know that the value of the card on position (o, b) is lower than or equal to the average value I know that o knew of the card previously on position (o, b) (Ax. 7.22). Similarly, I now know that agent o knows that the value of the card on position (o, b) is lower than or equal to the average value I know that o knew of the card previously on position (o, b) (Ax. 7.23).

$$K_iK_xK_o(card_{(o',b')} = \mathbf{f}) \rightarrow K_iK_x(V(card_{(o,b)}) \leq K_iK_xK_o\bar{V}(card_{(o',b')})) \tag{7.24}$$
$$K_iK_xK_o(card_{(o',b')} = \mathbf{f}) \rightarrow K_iK_xK_o(V(card_{(o,b)}) \leq K_iK_xK_o\bar{V}(card_{(o',b')})) \tag{7.25}$$

Similar to the previous rules, if I know that another agent x knows that agent o did not knew the card previously on position (o, b), then I know that agent x knows that the value of the card on position (o, b) is lower than or equal to the average value I know that agent x knows that agent o knew of the card previously on position (o, b) (Ax. 7.24). And, I now know that agent x knows that agent o knows that the value of the card on position (o, b) is lower than or equal to the average value I know that agent x knows that agent o knew of the card previously on position (o, b) (Ax. 7.25).

7.2.3 Update rules for both cases

$$K_iK_o((card_{(o',b')} = \mathbf{t}) \wedge (card_{(o,x\neq b)} = \mathbf{t})) \rightarrow K_i(V(card_{(o,x)}) \leq V(card_{open})) \tag{7.26}$$
$$K_iK_o((card_{(o',b')} = \mathbf{t}) \wedge (card_{(o,x\neq b)} = \mathbf{t})) \rightarrow K_iK_o(V(card_{(o,x)}) \leq V(card_{open})) \tag{7.27}$$

If I know that agent o knew the card previously on position (o, b), and I know that agent o knows one of his cards on position (o, x), where x is not b (so not the card just put), then I know that the value of the card on position (o, x) is lower than or equal to the value of the open pile card (Ax. 7.26). Also, I know that agent o knows that the value of the card on position (o, x) is lower than or equal to the value of the open pile card (Ax. 7.27).

$$K_iK_xK_o((card_{(o',b')} = \mathbf{t}) \wedge (card_{(o,y\neq b)} = \mathbf{t})) \rightarrow K_iK_x(V(card_{(o,y)}) \leq V(card_{open})) \quad (7.28)$$
$$K_iK_xK_o((card_{(o',b')} = \mathbf{t}) \wedge (card_{(o,y\neq b)} = \mathbf{t})) \rightarrow K_iK_xK_o(V(card_{(o,y)}) \leq V(card_{open})) \quad (7.29)$$

Similar to the previous rules, if I know that another agent x knows that agent o knew the card previously on position (o,b), and I know that agent x knows that agent o knows one of his cards on position (o,y), where y is not b, then I know that agent x knows that the value of the card on position (o,y) is lower than or equal to the value of the open pile card (Ax. 7.28). Also, I know that agent x knows that agent o knows that the value of the card on position (o,y) is lower than or equal to the value of the open pile card (Ax. 7.29).

$$K_iK_o((card_{(o',b')} = \mathbf{f}) \wedge (card_{(o,x\neq b)} = \mathbf{t})) \rightarrow K_i(V(card_{(o,x)}) \leq K_iK_o(\bar{V}(card_{(o',b')})) \quad (7.30)$$
$$K_iK_o((card_{(o',b')} = \mathbf{f}) \wedge (card_{(o,x\neq b)} = \mathbf{t})) \rightarrow K_iK_o(V(card_{(o,x)}) \leq K_iK_o(\bar{V}(card_{(o',b')})) \quad (7.31)$$

If I know that agent o did not knew the card previously on position (o,b), and I know that agent o knows one of his cards on position (o,x), where x is not b (so not the card just put), then I know that the value of the card on position (o,x) is lower than or equal to the average value I know that agent o knew of the card previously on position (o,b) (Ax. 7.30). Similarly, I now know that agent o knows that the value of the card on position (o,x) is lower than or equal to the average value I know that agent o knew of the card previously on position (o,b) (Ax. 7.31).

$$K_iK_xK_o((card_{(o',b')} = \mathbf{f}) \wedge (card_{(o,y\neq b)} = \mathbf{t})) \rightarrow K_iK_x(V(card_{(o,y)}) \leq K_iK_xK_o(\bar{V}(card_{(o',b')}))(7.32)$$
$$K_iK_xK_o((card_{(o',b')} = \mathbf{f}) \wedge (card_{(o,y\neq b)} = \mathbf{t})) \rightarrow K_iK_xK_o(V(card_{(o,y)}) \leq K_iK_xK_o(\bar{V}(card_{(o',b')}))(7.33)$$

Similar to the previous rules, if I know that another agent x knows that agent o did not knew the card previously on position (o,b), and I know that agent x knows that agent o knows one of his cards on position (o,y), where y is not b, then I know that agent x knows that the value of the card on position (o,y) is lower than or equal to the average value I know that agent x knows that agent o knew of the card previously on position (o,b) (Ax. 7.32). Likewise, I now know that agent x knows that agent o knows that the value of the card on position (o,y) is lower than or equal to the average value I know that agent x knows that agent o knew of the card previously on position (o,b) (Ax. 7.33).

8 General axioms

We have two general axioms which can be applied in all cases. These are triggered after the update rules have been applied for the corresponding action.

$$K_i\varphi \rightarrow K_iK_i\varphi \quad (8.1)$$

The first one corresponds to necessity. If I know φ, I know I know φ (Ax. 8.1).

$$K_iK_x\varphi \rightarrow K_iK_xK_x\varphi \quad (8.2)$$

This one is related to necessity. If I know that agent x knows φ, then I know that agent x knows that he knows that φ (Ax. 8.2).

9 Final remarks

With this list of axioms we probably haven't reached completeness of the model, however, we did show that our axiom system is sound. In addition, using this set of axioms our agents reason and apply knowledge with very good results.