The Intrinsic Quantum Nature of Nash Equilibrium Mixtures

Yohan Pelosse [1]

The concept of Nash equilibrium has become central in game theory, economics, and other social sciences. A Nash equilibrium is defined as an n-tuple of strategies or strategy profile (one strategy for each player) if each player's strategy is optimal against the others' strategies. As is well-known by now, the interactive epistemology under which rational individuals play such a "social equilibrium point" is quite demanding. Aumann and Brandenburger (1995) have notably demonstrated that it presumes that each player knows, or correctly guess his opponents' beliefs about the other players' strategy choices. In addition, as acknowledged in most of the literature, the notion of a mixed-strategy Nash equilibrium leads to some seemingly insuperable conceptual difficulties (see e.g., Aumann (1987)). The main trouble lies in the fact that, in a Nash equilibrium, each player who selects a mixed-strategy is always indifferent between two pure strategies of the support.

The purpose of this paper is to firm up the foundation of Nash equilibrium and provide a compelling (quantum) interpretation of this concept in the original rationalistic framework of Nash. The bulk of the paper is devoted to show how the Nash equilibrium notion can be constructively derived. The gist of our approach builds on the following two observations:

(i)*The classical game model is complete in the sense that its complete description is given by the strategy sets, the outcome map, and the payoff functions and*;

(ii)*Rationality is a relativistic or relational concept in the sense that it consists of making an optimal choice that has to be justifiable by some beliefs*.

Taken together, (i) and (ii) imply that absolute statements like A^i :="*strategy a is optimal in the game G for player i*" are generally neither absolutely "true", nor absolutely " false" but indeterminate. Hence, a non-classical logic—the three-valued logic of Lukasiewicz (1930)—enters the picture of the game model in its own right because this model does not (generally) contain the answers to questions like "*what constitutes a rational behavior?*".

So the question naturally arises: How will a player ascribe a relative truth-value, true, to a particular rational strategy?

The answer is simple; in the game model, each player possesses only pieces of a puzzle made of contingent statements about the optimality of a strategy. Playing

[1] Gate Lyon St-Etienne, 93 Chemin des mouilles 69130 Ecully (France).
Email address: yohanpelosse@googlemail.com

a Nash equilibrium is the picture on the box—the principle so to speak—to guide their assembly. Alternatively put, since absolute statements are generally neither true nor false, each player i who "speaks" a "rationalistic language" about rational strategies or beliefs can find the relative-truth values of contingent statements like "*strategy a is rational for player i*" is true if and only if "*each component of strategy profile b is (simultaneously) rational for each other player*" is true. [2] Of course, these tautologies correspond precisely to the Nash equilibria of the game. This is the central idea of the paper: The initial indeterminacy of what constitutes a rational strategy leads each player to break the Gordian Knot of endless chain of contingent statements by "self-interacting" in a Nash equilibrium. From this perspective, the Nash equilibrium notion must be seen as a "*rational determination principle*" followed by each player in order to unravel the initial indeterminacies of the game. This result is worth noting in view of the criticisms that have been made in the literature (see e.g., Bernheim (1984) and Pearce (1984)).

In the second part of the paper we are naturally led to ask: What are the conceptual and behavioral implications of this result on the interpretation of Nash equilibrium mixtures?

A Nash equilibrium has the advantage of existing in broad classes of games. However, existence results require the use of "mixed strategies" on the part of players. [3] In spite of the widespread use of these "probabilistic" equilibria, there is a small literature (see e.g., Harsanyi (1973), Aumann (1987) and Reny (2004)) and a much larger oral tradition arguing that there are conceptual problems for interpreting equilibrium mixtures. In order to respond to these criticisms, the second part of the paper examines the nature of equilibrium mixtures in games when we explicitly incorporate the indeterminism inherent to the rationality concept. We show that *before* the actual choice (of a pure strategy), the equilibrium state of mind of a player is described by the so-called "density matrices" of quantum mechanics (QM). Here, the density matrice represents the structure of knowledge of each player when he introspects himself (i.e. self-interacts) to determine his equilibrium strategy. We show that these density matrices are the inevitable consequence of the mental introspection of rational players. Finally, we prove that probabilities arising in an equilibrium mixture automatically satisfy the Born rule

[2] Hereafter, we will use the terms "relative" or "contingent", interchangeably.

[3] For example, Nash (1950) has proved its existence in finite strategic-form games. Glicksberg (1952) has proved existence when strategy spaces are non-empty and compact subset of a metric space and when payoff functions are continuous. More recently, Reny (1999), showed new results on the existence of mixed strategy Nash equilibria generalizing many existing conditions allowing for discontinuities in payoff functions.

of QM—one of the key postulates of QM—which demonstrates that equilibrium mixtures have a quantum origin. From a QM perspective, this rule which is responsible for practically all predictions of quantum physics has not been given a foundation from a first principle to date (see e.g., Landsman (2009)) As a result, these results unveil a deep connection between the mathematical formalism of QM and game theory.

Different streams of papers (see e.g., Eisert et al. (1999), Danilov and Lambert-Mogiliansky (2008) and Busemeyer and Lambert-Mogiliansky (2010)) have introduced formal tools of QM in game theory. This paper argues instead that classical game theory is already quantum-mechanical in nature. In a recent paper, Brandenburger (2010) establishes a formal connection between game theory and QM: He proves that adding quantum signals does not necessarily differ from the addition of classical signals. From this perspective, our results might be seen as a contribution to the rise of this literature. The next section provides a brief informal discussion of the main results of the paper.

Ontological conditions for Nash Equilibrium

In our first main result we show that the determination of a (intrinsic) Nash equilibrium, one for each player is the unavoidable consequence of rationality: Because of the indeterminism of what constitutes a rational strategy in a game, each player has to "self-interact" (in a consistent way) in order to break this initial indeterminism. We then have the following (Theorem 1)[4]: *Suppose we have a finite n-person game in strategic form where each player is rational and knows that the others are rational. Then σ is a Nash equilibrium profile in the game being played if and only each player determines the same intrinsic Nash equilibrium, σ.*

Theorem 1 shifts the usual interpretation of an equilibrium: Instead of describing the end-point of complex (social) strategic interactions, a Nash equilibrium explains the "internal" process of choice, wherein each individual self-interacts in a consistent way in order to determine his own rational behavior. Of course this accounts for the difference with Aumann and Brandenburger (1995)'s result which delineates the (tight) sufficient epistemic conditions leading each player to coordinate on the *same* equilibrium. From this perspective Theorem 1 lends support to the various interpretations of a Nash equilibrium as a norm of behavior or a focal

[4]In Theorem 1 we exclude the non-generic situations where a Nash equilibrium exists in strongly dominant strategy for some players and games where some players may use the same strategy in two different Nash equilibria. As shown in the formal analysis, this additional restriction can be easily dropped. Theorem 1 can also be extended to Euclidean games and some other usual classes of games with infinite strategy spaces.

point.[5]

The observation that rationality incorporates a non-Boolean logic is new. Hence, it is worth some discussion. The way of making logically precise the notion that a player behavior is not settled in a game is to use the three-valued logic of Lukasiewicz (1930). This non-Boolean logic captures the "ontological openness" of the players behavior in the game model i.e., the fact that the behavior is not settled in the game model until it is determined by the players themselves.[6] Hence, the interpretation of this logic is *ontic* here; the ontic (relative) sharp truth-values model the *act* of determination of what constitutes a rational behavior by a player in a game.

A natural interpretation of the hierarchies of beliefs is that players have an history or a "context" (see e.g., Brandenburger and Friendenberg (2008)). Of course, adding such a "context" would allow players to resolve the initial indeterminacy outright. But the original "rationalistic" game model does not a priori make such an assumption. Thus, from a methodological perspective, one could say that the presence of a non-Boolean logic marks the transition from an epistemic game theory (see e.g., Harsanyi (1973), Aumann (1987) and Brandenburger and Dekel (1993)) to an ontological analysis of game theory.

To the best of our knowledge, the Nash equilibrium concept has never been interpreted as a "self-interactive" solution concept. A notable exception is Perea (2007). In an epistemic model, Perea analyzes the classical game model from a single player's perspective by imposing conditions solely on the beliefs that a player has about the other players' strategy choices and the other players' beliefs i.e., he does not impose any "social" interactive epistemology. In this setting, Perea interprets a Nash equilibrium as describing the state of mind of a player. Theorem 1 confirms that this one-person interpretation is precisely the essence of the Nash equilibrium concept. From this perspective, Theorem 1 can be seen as the "ontological" complement of Perea.

The first part of the paper is concerned with the ontological condition for a Nash equilibrium. In the second part of the paper we aim at drawing the conceptual implications of this result. In particular we investigate the conceptual meaning of Nash equilibrium mixtures.

The quantum nature of Nash equilibrium mixtures

In the second main result of the paper we show that the "self-interaction" (in-

[5] See e.g., Schelling (1960), Binmore (2005) and Cubitt and Sugden (2011).

[6] Philosophically, this view is consistent with the idea that the truth of a proposition consists in representing an *actual* state of affairs.

trospection) of players induces each player to have a quantum state of mind.
(Theorem 2)[7]: *Suppose we have a finite n-person game in strategic form where each player is rational and knows that the others are rational. Then in any mixed Nash equilibrium profile $(\sigma^i)_{i\in N}$ of the game being played, the state of mind of each player i can be identified by a (unit) vector in a complex vector space, the state of mind of player i is fully described by an idempotent operator of trace 1 with complex off-diagonal terms*[8] *, and the empirical probability of having a pure strategy s^i in an experiment is given by the Born rule.*
Theorem 2 is a corollary of Theorem 1: By incorporing the initial indeterminism ingrained in the classical game model, Theorem 2 shows that the state-space structure and the probability rule—the so-called Born rule—of the textbook formulation of QM arises naturally in equilibrium. As an immediate corollary, when we incorporate the multiplicity of Nash equilibria in a game we obtain that the state space of possible (equilibrium global intrinsic) *mixed* states of a player is the space of *density matrices* (operators) of QM. Intuitively, a player may think that a pure strategy (in the support of the mixed strategy) is rational when looking at the game from his own perspective, but he may simultaneously think that another pure strategy is rational when he considers the (rational) common belief held by the other players at his other meta-perspective. This accounts for the fact that the state of mind of a player will always be given by a density matrix with some (complex) off-diagonal terms.[9] Hence, the heart of the quantum structure emerges naturally from two basic principles: The initial indeterminacy of what constitutes a rational behavior, together with its rational determination principle—the so-called Nash equilibrium concept.
Our results provide a compelling interpretation for the entries of the density matrices. Broadly speaking, we prove that these objects describe the knowledge structure built by players during their introspection. The off-diagonal terms of the density matrices—the so-called interferences in QM— turn out to be the hallmark of this (rational) "self-interaction". These interferences which appear in the so-called two slit experiment express the central puzzles of QM and are often considered to capture the essence of QM (Feynman et al. (1965)). There are indeed

[7]As for Theorem 1, this result can also be extended to Euclidean games and some other classes of games with infinite strategy spaces.

[8]It follows that, as in QM, the true intrinsic state space of a player is a complex projective space.

[9]Formally, the impossibility to have more than two perspectives is a consequence of Gleason's Theorem, while the impossibility to have a single perspective follows from the initial indeterminism of what constitutes a rational strategy.

numerous experimental evidences indicating that each individual particle (e.g., a photon in the classic two-slit experiment) can behave as if it were in different places at once. According to Theorem 2, if an observer were reading in the mind of a player before his choice, he would see the same quantum phenomena.[10]

Prima facie, the quantum nature of equilibrium mixtures in games seems to resolve the so-called "mixing problem". It says that the indifference condition is nothing but the expression of the quantum superposition of the player's states of mind between the different pure strategies in the support of his mixed strategy. In plain terms: Players are in a state of (rational) "indecision". This view certainly offers a resolution of the mixing problem before the actual choice (play) of the player. But it remains silent on how and when a player takes a particular pure strategy (in the support of his mixed strategy) during a measurement. Of course, this is the (unsolved) measurement problem of QM.

As in QM, the probabilities arising in a Nash equilibrium must therefore be seen as an expression of an "irreducible" randomness. This accounts for the intrinsic randomness of game theory: To put it in a nutshell, an outside observer cannot have more knowledge on the choice of the player than the player himself. From this viewpoint, the present approach can also be seen as a contribution to the growing literature on the information theoretic foundation of QM (see e.g., Zeilinger (1999), Fuchs and Schack (2011)).

Another important consequence of Theorem 2 is our derivation of the Born rule. Gleason's theorem gives both the state space of quantum mechanics and the probability rule but does not provide insight into the meaning of these postulates. In QM, recent attempts to derive the Born rule from more basic postulates of quantum theory include e.g., Deutsch (1999) and Landsman (2008). Deutsch is the first to study the emergence of QM probabilities within the non-probabilistic part of classical decision theory. The crucial difference with Deutsch's approach— and any other attempts to derive quantum axioms to date—is that in our case the *entire structure* of QM, together with the Born rule follows from a *single unavoidable principle*: The rational indeterminism ingrained in any system exhibiting an optimal behavior. [11] In Pelosse (2011a), we show that this "rational indetermination principle" implies that classical models of decision under uncertainty falls within the ambit of game theory and study the consequence thereof through the founda-

[10]In this experiment a stream of identical particles—all with the same speed and direction—is directed at a barrier with two slits. For an experimental evidence of the quantum nature of a *single* photon see e.g., Mandel (1999).

[11]In particular, Deutsch's framework requires the existence of some additional decision theoretic axioms, like the "*zero-sum rule*."

tion of random choice models. A thorough investigation of the implications of the present results for the foundations of QM is out of the scope of this paper. We do so, in a separate paper (Pelosse, 2011b) by analyzing the consequences of the "rational indetermination" of a single particle. There, we provide a game-theoretic account of the standard QM axioms, a justification for the von-Neumann algebra, derive the Heisenberg inequalities and show that the standard Schrödinger dynamics arise automatically.

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