Hybrid logics with infinitary proof systems

Barteld Kooi*
Gerard Renardel de Lavalette†
Rineke Verbrugge‡

University of Groningen
P.O. Box 800, 9700 AV Groningen, The Netherlands

Abstract

We provide a strongly complete infinitary proof system for hybrid logic. This proof system can be extended with countably many sequents. Thus, although these logics may be non-compact, strong completeness proofs are provided for infinitary hybrid versions of non-compact logics like ancestral logic and Segerberg's modal logic with the bounded chain condition. This extends the completeness result for hybrid logics by Gargov, Passy, and Tinchev.

Keywords: hybrid logic, strong completeness, non-compact logics, infinitary proof rules

1 Introduction

1.1 Hybrid logic

Hybrid logic is an extension of modal logic. Special propositional variables called *nominals*, which are true in exactly one possible world, are added to the language. Therefore they could equally be taken as names of possible worlds. Hybrid logic was initially developed by Prior in the 1960's [13], but there has been a flurry of activity surrounding hybrid logic in the past decade (see `www.hylo.net`). A textbook introduction to hybrid logic can be found in [2].

One of the pleasant features of hybrid logic is that its correspondence theory is very straightforward. Hybrid logic can be translated into first-order logic, where nominals are interpreted as constants. The link is so strong that it is very easy to obtain complete proof systems for classes of frames that satisfy additional properties. This works not only for the

*Department of Philosophy, `B.P.Kooi@rug.nl`
†Department of Computing Science, `G.R.Renardel.de.Lavalette@rug.nl`
‡Department of Artificial Intelligence, `rineke@ai.rug.nl`

usual properties such as transitivity, reflexivity, and symmetry, but also for irreflexivity, asymmetry, and many others that cannot be characterized by modal formulas. For example, the following axiom schema characterizes irreflexivity:

$$\mathsf{i} \to \neg\Diamond\mathsf{i}$$

The completeness theorem which is proved in [2] provides complete proof systems for many hybrid logics. It exploits the straightforward correspondence theory. When the base proof system is extended with *pure* axioms (axioms without propositional variables), then the new proof system is automatically complete for the class of corresponding frames.

1.2 Strong completeness

One would like to prove strong completeness also for logics where the relevant properties are not characterized by axioms, but by infinitary rules, such as the following rule, which characterizes the frame property that any state is reachable from any other state by a (finite) sequence of moves along the accessibility relation:

$$\{\neg @_{\mathsf{i}}\Diamond^n \mathsf{j} \mid n \in \mathbb{N}\} \vdash \bot$$

Note that, since there are countably many nominals in the language, this rule consists of countably many sequents. Although the completeness proof in [2] is very general, it is not applicable to non-compact modal logics, such as propositional dynamic logic (PDL), ancestral logic, other modal logics with (reflexive) transitive closure operators, and the 'reachability logic' given by the infinitary rule above. Moreover a finitary proof system cannot be strongly complete for such logics, since they are not compact. Therefore we focus on infinitary proof systems.

Let us remind the reader of some relevant definitions. *Strong completeness* (also called extended completeness) with respect to a class of frames S is the following property of a modal logical system S:

$$\Gamma \models_S \varphi \text{ implies } \Gamma \vdash_{\mathsf{S}} \varphi, \text{ for all formulas } \varphi \text{ and all sets of formulas } \Gamma.$$

This generalizes weak completeness, where Γ is empty. Observe that weak completeness implies strong completeness whenever the logic in question is *semantically compact*, i.e. when $\Gamma \models_S \varphi$ implies that there is a finite $\Gamma' \subseteq \Gamma$ with $\Gamma' \models_S \varphi$, hence $\models_S \bigwedge \Gamma' \to \varphi$. This is, for example, the case in modal logics such as K and S5.

1.3 Another modal logic: PDL

Propositional dynamic logic (PDL) is a well-known example of a non-compact logic. PDL is a modal logic with modal operators for atomic programs (a), and more complex programs that are constructed using sequential composition ($\alpha;\beta$), nondeterministic choice ($\alpha \cup \beta$), and iteration (α^*). For a textbook introduction see [8]. We have for the relevant class of frames S, that $\{[a^n]p \mid n \in \mathbb{N}\} \models_S [a^*]p$ but there is no natural number k with $\{[a^n]p \mid n \leq k\} \models_S [a^*]p$. As a consequence, we do not have strong completeness for any finitary axiomatization, *a fortiori* not for its usual, weakly complete proof system. So strong completeness requires an infinitary proof system. Here, "infinitary" does not refer to the language (all formulas in this paper have finite length) nor to the derivations (there are no infinite branches in the derivations), but to the *derivation relation* (proof sequents may be non-standard in requiring infinitely many premises). Although all branches are finite, it can still be the case that there is no uniform bound on the length of the branches of a given derivation. The usual transfinite induction principles on proofs remain valid.

Infinitary non-hybrid versions of such non-compact modal logics were investigated and strong completeness proofs were given by Goldblatt [7], Segerberg [16] and the present authors [14]. In those cases a strongly complete infinitary proof system can be obtained by adding infinitary rules: one simply makes a rule from an example that shows non-compactness. In [12], Passy and Tinchev investigate hybrid versions of PDL and present an infinitary proof system for the language with nominals and the universal modality, which is shown to be strongly complete.

1.4 Main goal

The main goal of this paper is to extend the completeness result for PDL obtained in [14] and [15] (an updated version of [14] with a shorter completeness proof) to hybrid logic, thus also extending the completeness result in [2] for hybrid logic to hybrid systems that are not semantically compact. This generalization is required for the characterization of interesting frame properties (see section 5). Rather than extending the base system of hybrid logic with pure axioms, we allow extensions with countably many pure *sequents*, each with possibly infinitely many premises. We will first prove a general result about a basic hybrid system extended with a countable set of sequents; then we show some applications to hybrid versions of specific modal logics. Globally speaking, the method of proof used here is the same as in [14, 15]: a Lindenbaum lemma is used to extend a consistent set to a saturated set, and the saturated set is used to construct a canonical model. The details are rather different, however. In [15], we show that every saturated set is maximal consistent, and construct a canonical model from maximal consistent sets as usual, which is uncountable in general. In this paper, we use one saturated set to define an equivalence relation on nominals and obtain a countable canonical model with equivalence classes of nominals as worlds. Nominals (names for worlds) are present in hybrid logic, but not in PDL. So nominals do make life somewhat easier when it comes to proving completeness.

In Section 2 we briefly introduce the language and semantics of hybrid logic. In Section 3 we provide the infinitary proof system for hybrid logic. In Section 4 we show this proof system is complete. In Section 5 we discuss some specific extensions of the basic proof system. In Section 6 we discuss related literature. Finally, in Section 7 we draw conclusions and indicate directions for further research.

2 Language and semantics

There are some variants of the language of hybrid logic. In this paper, we work with the extension of the language of modal logic with nominals i, *at*-operators $@_{\mathsf{i}}$ and nominal binders $\downarrow\mathsf{i}$. The nominal binder $\downarrow$ is not essential for our arguments, and we only included it in our definition of hybrid logic for reasons of generality. We also find the axiomatization with $\downarrow$ slightly more elegant than the axiomatization without it, which contains the Name Rule and the Paste Rule. We come back to this issue in the concluding section 7.

Observe that the language is finitary, i.e. all formulas are finite: only the sequents in the proof system are possibly infinite (see the next section).

Definition 1 (Language of hybrid logic)
Let a countable set of propositional variables P, and a countably infinite set of nominals I be given. The language of hybrid logic $\mathcal{L}(P, \mathsf{I})$ is given by the following BNF:

$$\varphi ::= \bot \mid p \mid \mathsf{i} \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Box\varphi \mid @_{\mathsf{i}}\varphi \mid \downarrow\mathsf{i}\varphi$$

where $p \in P$, and $\mathsf{i} \in \mathsf{I}$. We use the usual abbreviations for $\vee, \rightarrow, \leftrightarrow$ and $\Diamond$. We are usually sloppy and write $\mathcal{L}$ instead of $\mathcal{L}(P, \mathsf{I})$. $\Box$

A formula $@_{\mathsf{i}}\varphi$ is to be read as "φ holds at the world named i", and $\downarrow\mathsf{i}\varphi$ as "φ holds when i is interpreted as the actual world". In $\downarrow\mathsf{i}\varphi$, $\downarrow\mathsf{i}$ binds all free occurrences of i in φ.

The functions $\mathsf{nom}, \mathsf{fnom} : \mathcal{L} \rightarrow \wp(\mathsf{I})$ yield the set of nominals, respectively the set of free nominals, that occur in a formula. They are defined inductively as expected, and differ only in the clause for $\downarrow$:

$$\begin{array}{rcl} \mathsf{nom}(\downarrow\mathsf{i}\varphi) & = & \mathsf{nom}(\varphi) \cup \{\mathsf{i}\} \\ \mathsf{fnom}(\downarrow\mathsf{i}\varphi) & = & \mathsf{fnom}(\varphi) - \{\mathsf{i}\} \end{array}$$

We generalize nom and fnom to sets of formulas, and later to proof sequents and proofs.

The models used in the semantics of hybrid logic are models for modal logic extended with a valuation for the nominals.

Definition 2 (Models for hybrid logic)
A model for $\mathcal{L}$ is a quadruple $M = (W, R, V, A)$ such that:

- $W \neq \emptyset$; a set of possible worlds;
- $R \subseteq W \times W$; an accessibility relation;
- $V : P \rightarrow \wp(W)$; assigns a set of possible worlds to each propositional variable;
- $A : \mathsf{I} \rightarrow W$; assigns a possible world to each nominal.

A frame F is a tuple (W, R), where W and R are as above. $\square$

Nominal assignments A can be changed locally: if i is a nominal and $w \in W$, then $A' = A[\mathsf{i} := w]$ behaves like A on $\mathsf{I} - \{\mathsf{i}\}$, and $A'(\mathsf{i}) = w$. We extend this to models: $(W, R, V, A)[\mathsf{i} := w] = (W, R, V, A[\mathsf{i} := w])$.

Definition 3 (Named model)
A model $M = (W, R, V, A)$ for $\mathcal{L}$ is *named* if every world in the model is the denotation of a nominal, i.e. A is a surjective function of I onto W. $\square$

Observe that a named model is always countable.

Definition 4 (Semantics)
Let a model $M = (W, R, V, A)$ and $w \in W$ be given. Let $p \in P$, $\mathsf{i} \in \mathsf{I}$, and $\varphi, \psi \in \mathcal{L}$.

$$\begin{array}{lll}
(M, w) \not\models \bot & & \\
(M, w) \models p & \text{iff} & w \in V(p) \\
(M, w) \models \mathsf{i} & \text{iff} & A(\mathsf{i}) = w \\
(M, w) \models \neg\varphi & \text{iff} & (M, w) \not\models \varphi \\
(M, w) \models \varphi \wedge \psi & \text{iff} & (M, w) \models \varphi \text{ and } (M, w) \models \psi \\
(M, w) \models \Box\varphi & \text{iff} & (M, v) \models \varphi \text{ for all } v \text{ with } (w, v) \in R \\
(M, w) \models @_{\mathsf{i}}\varphi & \text{iff} & (M, A(\mathsf{i})) \models \varphi \\
(M, w) \models \downarrow\!\mathsf{i}\varphi & \text{iff} & (M[\mathsf{i} := w], w) \models \varphi
\end{array}$$

Given a set of formulas Γ we write $(M, w) \models \Gamma$ iff $(M, w) \models \varphi$ for every $\varphi \in \Gamma$. We write $\Gamma \models \varphi$ iff $(M, w) \models \Gamma$ implies $(M, w) \models \varphi$ for every model M and world w. We write $M \models \varphi$ iff $(M, w) \models \varphi$ for every world w. We write $M \models \Gamma/\varphi$ iff $(M, w) \models \Gamma$ implies $(M, w) \models \varphi$ for every world w. Given a frame F and a world w, we say that $(F, w) \models \varphi$ iff $((F, V, A), w) \models \varphi$ for every pair of valuations V, A on F.

Likewise, we write $(F, w) \models \Gamma$ iff $(F, w) \models \varphi$ for every $\varphi \in \Gamma$; we write $F \models \varphi$ iff $(F, w) \models \varphi$ for every world w; and we write $F \models \Gamma/\varphi$ iff for every pair of valuations V, A and every $w \in W$, $((F, V, A), w) \models \Gamma$ implies $((F, V, A), w) \models \varphi$.

Note that the notion $M \models \Gamma/\varphi$, being defined locally, is much stronger than the notion "if $M \models \Gamma$, then $M \models \varphi$"; similarly for $F \models \Gamma/\varphi$. $\square$

Let $M = (W, R, V, A)$, $M' = (W', R', V', A')$ be two models of $\mathcal{L}(P, \mathsf{I})$ and let $\mathsf{J} \subseteq \mathsf{I}$. We say that M and M' are J-equivalent (notation: $M =_{\mathsf{J}} M'$) iff $W = W', R = R', V = V'$ and $A(\mathsf{j}) = A'(\mathsf{j})$ for all $\mathsf{j} \in \mathsf{J}$. We have

Lemma 1
If $M =_{\mathsf{fnom}(\varphi)} M'$, then $(M, w) \models \varphi \Leftrightarrow (M', w) \models \varphi$ □

3 The proof system Khyb_ω

The proof system is based on *sequents*, i.e. expressions of the form $\Gamma \vdash \varphi$ where Γ is a (possibly infinite) collection of formulas. Observe that Γ is always countable, since the number of formulas in the language is countable.

We shall restrict ourselves to sequents $\Gamma \vdash \varphi$ where $(\mathsf{I} - \mathsf{nom}(\Gamma, \varphi))$ is infinite (I being the countably infinite collection of nominals of the language). This is a mild restriction: by nominal renaming, any sequent not satisfying the restriction can be transformed into a sequent satisfying it. The reason for the restriction is that it yields the Fresh Nominal Principle:

> there is always a fresh (i.e. unused) nominal.

This principle allows us to (repeatedly) pick a fresh nominal from the set of nominals not being used yet. We shall do this e.g. in the proof of the Paste Rule in Lemma 3 and in the construction of the maximal finitely consistent set Δ_ω in the completeness proof (Theorem 2). This approach contrasts with the usual definition of the infinitary logic $\mathcal{L}_{\omega_1\omega}$ (see [9]), where the Fresh Variable Principle (a variant of our Fresh Nominal Principle) is realized by adopting an uncountable collection of variables in the language definition. Note that the Fresh Nominal Principle is reminiscent of many constructions in the history of logic. For a recent example, see the 'freshness quantifier' of [5].

Before presenting the proof system, we define substitution.

Definition 5 (Substitution)
A substitution is a function $\sigma : \mathsf{I} \to \mathsf{I}$. Substitutions can be adapted locally: if σ is a substitution and $\mathsf{i}, \mathsf{j} \in \mathsf{I}$, then $\sigma' = \sigma[\mathsf{i} := \mathsf{j}]$ is the substitution that behaves like σ on $\mathsf{I} - \{\mathsf{i}\}$, and $\sigma'(\mathsf{i}) = \mathsf{j}$.
Application $\varphi\sigma$ of substitution σ to formula φ is defined as expected: the only nontrivial clause is

$$(\downarrow\mathsf{i}\varphi)\sigma = \downarrow\mathsf{j}(\varphi(\sigma[\mathsf{i} := \mathsf{j}]))$$

where $\mathsf{j} \notin \sigma[\mathsf{nom}(\varphi)] = \{\sigma(\mathsf{k}) \mid \mathsf{k} \in \mathsf{nom}(\varphi)\}$. The specific choice of j is irrelevant, since we have, for all $\mathsf{j}_1, \mathsf{j}_2 \notin \{\sigma(\mathsf{k}) \mid \mathsf{k} \in \mathsf{nom}(\varphi)\}$ that

$$\models \downarrow\mathsf{j}_1(\varphi[\mathsf{i} := \mathsf{j}_1]) \leftrightarrow \downarrow\mathsf{j}_2(\varphi[\mathsf{i} := \mathsf{j}_2]).$$

Lemma 2 (Substitution property)
Let $M = (W, R, V, A)$ be a model and σ a substitution. Define M_σ by $M_\sigma = (W, R, V, A \circ \sigma)$. Then we have

$$(M, w) \models \varphi\sigma \Leftrightarrow (M_\sigma, w) \models \varphi$$

Proof Formula induction, using Lemma 1 and the property: if σ, τ are two substitutions with $\sigma(\mathsf{i}) = \tau(\mathsf{i})$ for all $\mathsf{i} \in \mathsf{fnom}(\varphi)$, then $(M, w) \models \varphi\sigma$ iff $(M, w) \models \varphi\tau$. $\square$

Definition 6 (Proof system for hybrid logic)
The proof system consists of the *axiom sequents* and *sequent rules* provided in Figure 1. Derivability is defined inductively as usual: a sequent is derivable when it is an axiom, or when it is the conclusion of a rule with derivable sequents as premises. Observe that, due to the infinitary cut rule, derivations may contain infinitely many sequents. For $* \in \{\Box, @_\mathsf{i}, \downarrow\mathsf{i} \mid \mathsf{i} \in \mathsf{I}\}$, we write $*\Gamma$ for $\{*\varphi \mid \varphi \in \Gamma\}$; moreover, $\Gamma \vdash \Delta$ denotes ($\Gamma \vdash \varphi$ for every $\varphi \in \Delta$). $\square$

Our axioms are virtually the same as in the axiomatization of the finitary hybrid logic $\mathcal{H}(@, \downarrow)$ in [17, Chapter 9]: the main difference is that Ten Cate uses the equivalent formulation $\vdash @_\mathsf{i}(\dots)$ instead of $\mathsf{i} \vdash \dots$ for **DA** and **BG**. (A minor difference of our approach with [17] is that we do not introduce a syntactical category for bound nominals.) Our infinitary sequent rules for strong necessitation are straightforward generalizations of the corresponding finitary rules.

Theorem 1 (Soundness)
The proof system is sound: if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$. $\square$

Proof As usual, by transfinite induction on the length of a derivation of $\Gamma \vdash \varphi$. We treat a few typical cases.
$\models$ **DA**, i.e. $\mathsf{i} \models \downarrow\mathsf{j}\varphi \leftrightarrow \varphi[\mathsf{j} := \mathsf{i}]$ reads: for all models M and worlds w

$$w = A(\mathsf{i}) \Rightarrow ((M[\mathsf{j} := w], w) \models \varphi \Leftrightarrow (M[\mathsf{j} := w], w) \models \varphi[\mathsf{j} := \mathsf{i}])$$

and this follows from Lemma 2 and the fact that

$$w = A(\mathsf{i}) \Rightarrow M[\mathsf{j} := w]_{[\mathsf{j}:=\mathsf{i}]} = M[\mathsf{j} := w]$$

$\models$ **Name**, i.e. $\models \downarrow\mathsf{i}@_\mathsf{i}\varphi \rightarrow \varphi$ means that, for all M, w:

$$(M[\mathsf{i} := w], w) \models \varphi \Rightarrow (M, w) \models \varphi$$

Since $\mathsf{i} \notin \mathsf{fnom}(\varphi)$, we have $M[\mathsf{i} := w] =_{\mathsf{fnom}(\varphi)} M$; now apply Lemma 1.
$\models$ **BG**, i.e. $\mathsf{i} \models \Box \downarrow\mathsf{j}@_\mathsf{i}\Diamond\mathsf{j}$ boils down to: for all M, w

$$A(\mathsf{i}) = w \Rightarrow \forall v((w, v) \in R \Rightarrow \exists u((A[\mathsf{j} := v](\mathsf{i}), u) \in R \;\&\; A[\mathsf{j} := v](\mathsf{j}) = u)$$

since $\mathsf{i} \neq \mathsf{j}$ and $A(\mathsf{i}) = w$ imply that $A[\mathsf{j} := v](\mathsf{i}) = w$, this follows from $\forall v((w, v) \in R \Rightarrow \exists u((w, u) \in R \;\&\; v = u))$, which is true. $\square$

Taut	$\vdash \varphi$ if φ is an instance of a propositional tautology	
MP	$\varphi, \varphi \to \psi \vdash \psi$	(modus ponens)
$\mathbf{K}_\Box$	$\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	(distribution)
$\mathbf{K}_@$	$\vdash @_\mathsf{i}(\varphi \to \psi) \to (@_\mathsf{i}\varphi \to @_\mathsf{i}\psi)$	(distribution)
$\mathbf{SD}_@$	$\vdash @_\mathsf{i}\varphi \to \neg @_\mathsf{i}\neg\varphi$	(self-dual)
Intr	$\vdash \mathsf{i} \wedge \varphi \to @_\mathsf{i}\varphi$	(introduction)
$\mathbf{T}_@$	$\vdash @_\mathsf{i}\mathsf{i}$	(reflexivity)
Agree	$\vdash @_\mathsf{i}@_\mathsf{j}\varphi \leftrightarrow @_\mathsf{j}\varphi$	(agree)
Back	$\vdash \Diamond @_\mathsf{i}\varphi \to @_\mathsf{i}\varphi$	(back)
DA	$\mathsf{i} \vdash \downarrow\mathsf{j}\varphi \leftrightarrow \varphi[\mathsf{j} := \mathsf{i}]$	(downarrow)
Name	$\vdash \downarrow\mathsf{i}@_\mathsf{i}\varphi \to \varphi$, provided $\mathsf{i} \notin \mathsf{fnom}(\varphi)$	(name)
BG	$\mathsf{i} \vdash \Box \downarrow\mathsf{j}@_\mathsf{i}\Diamond\mathsf{j}$ provided $\mathsf{i} \neq \mathsf{j}$	(bounded generalization)
$\mathbf{SNec}_\Box$	if $\Gamma \vdash \varphi$, then $\Box\Gamma \vdash \Box\varphi$	(strong necessitation)
$\mathbf{SNec}_@$	if $\Gamma \vdash \varphi$ then $@_\mathsf{i}\Gamma \vdash @_\mathsf{i}\varphi$	(strong necessitation)
$\mathbf{SNec}_\downarrow$	if $\Gamma \vdash \varphi$ then $\downarrow\mathsf{i}\Gamma \vdash \downarrow\mathsf{i}\varphi$	(strong necessitation)
InfCut	if $\Gamma \vdash \Delta$ and $\Gamma', \Delta \vdash \varphi$ then $\Gamma, \Gamma' \vdash \varphi$	(infinitary cut)
Ded	if $\Gamma, \varphi \vdash \psi$ then $\Gamma \vdash \varphi \to \psi$	(deduction)

Figure 1: The axiom sequents and sequent rules of Khyb_ω

Before we go on, we derive some sequents and proof rules.

Lemma 3
The following sequents and rules are derivable:

W	if $\Gamma \vdash \varphi$ then $\Gamma, \Delta \vdash \varphi$
$\mathbf{B}_@$	$\vdash @_i j \leftrightarrow @_j i$
Nom	$\vdash @_i j \wedge @_j \varphi \rightarrow @_i \varphi$
Bridge	$\Diamond i, \Box \varphi \vdash @_i \varphi$
$\mathbf{SD}_\downarrow$	$\vdash \downarrow i \neg \varphi \rightarrow \neg \downarrow i \varphi$
Namerev	$\vdash \varphi \rightarrow \downarrow i @_i \varphi$ provided $i \notin \mathsf{fnom}(\varphi)$
NR	if $\Gamma, i \vdash \varphi$ then $\Gamma \vdash \varphi$, provided $i \notin \mathsf{fnom}(\Gamma, \varphi)$
VB	$\vdash \varphi \leftrightarrow \downarrow i \varphi$ provided $i \notin \mathsf{fnom}(\varphi)$
PR	if $\Gamma, @_i \Diamond j \vdash @_j \varphi$ then $\Gamma \vdash @_i \Box \varphi$, provided $j \notin \mathsf{fnom}(\Gamma, \varphi) \cup \{i\}$

Proof **W** (weakening) follows directly from **InfCut**, by taking $\Delta = \emptyset$. $\mathbf{B}_@$, **Nom** and **Bridge** are well known consequences of the other axioms on nominals, which are obtained as follows.
$\mathbf{B}_@$: by **Intr**, $\mathbf{SNec}_@$, **Ded** and propositional reasoning, we have $\vdash @_i j \wedge @_i i \rightarrow @_i @_j i$; with $\mathbf{T}_@$, **Agree** and **MP**, this yields $@_i j \rightarrow @_j i$. The other direction is proved similarly.
Nom: by **Intr** and contraposition plus $\mathbf{SD}_@$, we have $\vdash j \wedge @_j \varphi \rightarrow \varphi$; with $\mathbf{SNec}_@$ and **Agree**, we now get $\vdash @_i j \wedge @_j \varphi \rightarrow @_i \varphi$, i.e. **Nom**.
Bridge: by modal reasoning, we have $\Diamond i, \Box \varphi \vdash \Diamond(i \wedge \varphi)$; also (by **Intr**, $\mathbf{K}_\Box$, contraposition and **Back**) we have $\Diamond(i \wedge \varphi) \vdash @_i \varphi$, so we get $\Diamond i, \Box \varphi \vdash @_i \varphi$.

The derivation of the Name rule **NR** and the Paste rule **PR** is more involved: we use $\mathbf{SD}_\downarrow$, **Namerev** and **VB** (vacuous binding) as intermediate results.
$\mathbf{SD}_\downarrow$: by **DA** and propositional reasoning, we have $i \vdash \downarrow i \neg \varphi \leftrightarrow \neg \varphi$ and $i \vdash \neg \downarrow i \varphi \leftrightarrow \neg \varphi$, so $i \vdash \downarrow i \neg \varphi \leftrightarrow \neg \downarrow i \varphi$. With $\mathbf{SNec}_@$ and $\mathbf{T}_@$, we get $\vdash @_i(\downarrow i \neg \varphi \leftrightarrow \neg \downarrow i \varphi)$; via $\mathbf{SNec}_\downarrow$, **Name** and propositional reasoning, we obtain $\vdash \downarrow i \neg \varphi \leftrightarrow \neg \downarrow i \varphi$.
Namerev: let $i \notin \mathsf{fnom}(\varphi)$, then (by **Name** and propositional reasoning) $\vdash \varphi \rightarrow \neg \downarrow i @_i \neg \varphi$, and $\vdash \varphi \rightarrow \downarrow i @_i \varphi$ now follows via $\mathbf{SD}_@$ and $\mathbf{SD}_\downarrow$.
NR: if $\Gamma, i \vdash \varphi$, then (via $\mathbf{SNec}_@$, $\mathbf{T}_@$ and propositional reasoning) we have $@_i \Gamma \vdash @_i \varphi$ and (via $\mathbf{SNec}_\downarrow$) $\downarrow i @_i \Gamma \vdash \downarrow i @_i \varphi$; by **Name** and **Namerev**, we now obtain $\Gamma \vdash \varphi$.
VB: let $i \notin \mathsf{fnom}(\varphi)$; by **DA**, we have $i \vdash \downarrow i \varphi \leftrightarrow \varphi$, so with **NR** we obtain $\vdash \downarrow i \varphi \leftrightarrow \varphi$.
PR: let $j \notin \mathsf{fnom}(\Gamma, \varphi) \cup \{i\}$. If $\Gamma, @_i \Diamond j \vdash @_j \varphi$ then (by $\mathbf{SNec}_@$ and **Agree**) we have $@_k \Gamma, @_i \Diamond j \vdash @_j \varphi$ for some fresh k. With $\mathbf{SNec}_\downarrow$, we get $\downarrow j @_k \Gamma, \downarrow j @_i \Diamond j \vdash \downarrow j @_j \varphi$; by **VB** and **Name**, we obtain $@_k \Gamma, \downarrow j @_i \Diamond j \vdash \varphi$.

Via $\mathbf{SNec}_\Box$ and $@_\mathsf{k}\Gamma \vdash \Box @_\mathsf{k}\Gamma$ (this is a consequence of **Back**) we now get $@_\mathsf{k}\Gamma, \Box\downarrow\mathsf{j}@_\mathsf{i}\Diamond\mathsf{j} \vdash \Box\varphi$, and (via $\mathbf{SNec}_@$ and **Agree**) $@_\mathsf{k}\Gamma, @_\mathsf{i}\Box\downarrow\mathsf{j}@_\mathsf{i}\Diamond\mathsf{j} \vdash @_\mathsf{i}\Box\varphi$. With **BG** we have $@_\mathsf{k}\Gamma \vdash @_\mathsf{i}\Box\varphi$, so via **Intr** $\Gamma, \mathsf{k} \vdash @_\mathsf{i}\Box\varphi$ and finally with **NR** we obtain $\Gamma \vdash @_\mathsf{i}\Box\varphi$.

□

4 Strong completeness of Khyb_ω plus countably many sequents

Take the basic system Khyb_ω defined above, and add a *denumerable* set of additional axiom sequents:

$$\mathbf{AS} = \{\Gamma_n \vdash \varphi_n \mid n \in \mathbb{N}\}$$

What **AS** contains may depend on the language, i.e. the parameters P and I. E.g., **AS** may contain (or even consist of) sequents generated by substitution from sequent schemata. When we add all instances of a sequent $\Gamma \vdash \varphi$ to **AS** that are obtained by arbitrary substitutions of formulas for propositional variables and nominals for nominals, then countability is only guaranteed if $\Gamma \vdash \varphi$ is parameter-finite, i.e. if $\Gamma \vdash \varphi$ contains only finitely many nominals and propositional variables. Note, incidentally, that the following rule, which characterizes converse well-foundedness of R, is *not* a countable set of sequents:

CWF $\{@_{\mathsf{i}_n}\Diamond\mathsf{i}_{n+1} \mid n \in \mathbb{N}\} \vdash \perp$ where $\mathsf{i}_0, \mathsf{i}_1, \ldots$ is an arbitrary sequence of nominals

In this section we provide a strong completeness proof for $\mathsf{Khyb}_\omega + \mathbf{AS}$. Our proof is inspired by the completeness proofs for the infinitary logic $\mathcal{L}_{\omega_1\omega}$ in [9], the strong completeness proof for infinitary modal logics in [7], and the completeness proof for finitary hybrid logic presented in [2]. This last proof is very general and also shows that when the proof system is extended with extra pure axioms, then this extended proof system is automatically strongly complete with respect to the class of frames defined by these pure axioms [6]. However, it is a finitary proof system, and the completeness proof hinges on a Lindenbaum Lemma where compactness is assumed. So if we were to add an infinitary rule to their system, we would not get a complete proof system. Therefore we borrowed ideas from the completeness proofs in [9] and [7], which do not depend on compactness. Furthermore we show that extensions of Khyb_ω with extra pure axiom sequents with finitely many nominals are also complete for the class of frames defined by these rules.

Theorem 2 (Strong completeness)
Every $\mathsf{Khyb}_\omega + \mathbf{AS}$-consistent set of formulas in language $\mathcal{L}$ is satisfiable in a named model in which **AS** is valid. □

Proof The proof has the usual structure. Given a consistent set Γ, we extend the language with countably many fresh nominals, show that Γ is still consistent in the new language, and extend Γ to a maximal finitely consistent set Δ_ω satisfying some additional properties. Δ_ω is used to define a named model M, in which every world is an equivalence class of nominals. For M we prove a truth lemma, relating validity in M to being an element of Δ_ω. As a consequence we have that, in some world in M, every formula of Γ holds. Moreover, all axiom sequents of **AS** are valid in M.

So assume Γ is consistent, i.e. $\Gamma \nvdash \bot$. We extend the language $\mathcal{L} = \mathcal{L}(P, \mathsf{I})$ to $\mathcal{L}^+ = \mathcal{L}(P, \mathsf{J})$, where $\mathsf{J} \supseteq \mathsf{I}$ with $\mathsf{J} - \mathsf{I}$ infinite. (Consequently, **AS** may now also be extended.) We claim $\Gamma \nvdash_{\mathcal{L}^+} \bot$. For assume $\Gamma \vdash_{\mathcal{L}^+} \bot$ and let Π^+ be the $\mathcal{L}^+$-derivation of this sequent. We shall show that Π^+ can be transformed into a $\mathcal{L}$-derivation Π of $\Gamma \vdash \bot$, contradicting our assumption. Let

$$c : (\mathsf{nom}(\Pi^+) - \mathsf{nom}(\Gamma)) \to (\mathsf{I} - \mathsf{nom}(\Gamma))$$

be an injection (such an injection exists, for $(\mathsf{I} - \mathsf{nom}(\Gamma))$ is infinite). Now replace in Π^+ all (free and bound) nominals $\mathsf{i} \in \mathsf{nom}(\Pi^+) - \mathsf{nom}(\Gamma)$ by $c(\mathsf{i})$: this yields the $\mathcal{L}$-derivation Π.

A collection $\Delta \subseteq \mathcal{L}^+$ is called *admissible* if it is $\mathcal{L}^+$-consistent and $\mathsf{J} - \mathsf{nom}(\Delta)$ is infinite. We shall define an increasing sequence $\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_n \subseteq \ldots$ of admissible sets containing Γ, and use $\Delta_\omega = \bigcup_n \Delta_n$ to construct the model. Let $\mathsf{i} \in \mathsf{J} - \mathsf{I}$. We put $\Delta_0 = \Gamma \cup \{\mathsf{i}\}$, so Δ_0 is indeed admissible (consistency follows from **NR**).

Now let $\{\varphi_n \mid n \in \omega\}$ be an enumeration of $\mathcal{L}^+$ where every formula occurs infinitely often. This yields an infinite set of numbers $N_\varphi = \{n \mid \varphi_n = \varphi\}$ for every formula $\varphi \in \mathcal{L}^+$. Let $\{\Theta_n \vdash \psi_n \mid n \in \mathbb{N}\}$ be an enumeration of $@\mathbf{AS} = \{@_\mathsf{j}\Theta \vdash @_\mathsf{j}\psi \mid (\Theta \vdash \psi) \in \mathbf{AS}, \mathsf{j} \in \mathsf{J}\}$; and let $M_\varphi = \{m \mid (\Theta_m \vdash \varphi) \in @\mathbf{AS}\}$ be a (possibly empty) set of numbers for every formula φ. We define for each φ an *injective* function $f_\varphi : M_\varphi \to N_\varphi$ such that $f_\varphi(m_k) = n_k$ (i.e. the k-th number in M_φ is mapped to the k-th number in N_φ).

Now we define Δ_{n+1} in terms of Δ_n, and show that Δ_{n+1} is admissible if Δ_n is. We distinguish between $\Delta_n \vdash \varphi_n$ and $\Delta_n \nvdash \varphi_n$.
$\Delta_n \vdash \varphi_n$: define Δ_{n+1} as follows, where $\mathsf{j} \in \mathsf{J} - \mathsf{nom}(\Delta_n, \varphi_n)$ (this set is nonempty, by admissibility of Δ_n).

$$\Delta_{n+1} \quad = \quad \begin{cases} \Delta_n \cup \{\varphi_n, @_\mathsf{k}\Diamond\mathsf{j}, @_\mathsf{j}\neg\psi\} & \text{if } \varphi_n = @_\mathsf{k}\neg\Box\psi \\ \Delta_n \cup \{\varphi_n\} & \text{otherwise} \end{cases}$$

We claim that Δ_{n+1} is admissible: this comes down to showing that it is consistent. For the last clause, this is evident, so we concentrate on the first clause. Assume, to the contrary, that Δ_{n+1} is inconsistent, so $\Delta_n, @_\mathsf{k}\neg\Box\psi, @_\mathsf{k}\Diamond\mathsf{j}, @_\mathsf{j}\neg\psi \vdash \bot$. Hence (using $\mathbf{SD}_@$) $\Delta_n, @_\mathsf{k}\neg\Box\psi, @_\mathsf{k}\Diamond\mathsf{j} \vdash @_\mathsf{j}\psi$; with **PR** (recall that j is fresh), we get $\Delta_n, @_\mathsf{k}\neg\Box\psi \vdash @_\mathsf{k}\Box\psi$, which yields

contradiction with the consistency of Δ_n and $\Delta_n \vdash @_{\mathsf{k}}\neg\Box\psi$.
$\Delta_n \nvdash \varphi_n$: now we define

$$\Delta_{n+1} \quad = \quad \begin{cases} \Delta_n \cup \{\neg\varphi_n, \neg\theta\} & \text{if there is an } m \text{ with } f_{\varphi_n}(m) = n \\ & \text{(i.e. there exists a } (\Theta_m, \varphi_n) \in @\mathbf{AS}) \\ & \text{and } \theta \text{ is the smallest formula in } \Theta_m \\ & \text{such that } \Delta_n \cup \{\neg\varphi_n, \neg\theta\} \text{ is consistent.} \\ \Delta_n \cup \{\neg\varphi_n\} & \text{otherwise} \end{cases}$$

We claim that the definition is correct, i.e. that, in the first clause, such a θ can always be found: for if not, then we would have $\Delta_n, \neg\varphi_n \vdash \theta$ for all $\theta \in \Theta_m$ and hence (since $@\mathbf{AS}$ follows from $\mathbf{AS}$ by rule $\mathbf{SN}_@$) we would have $\Delta_n, \neg\varphi_n \vdash \varphi_n$, contradicting $\Delta_n \nvdash \varphi_n$. The admissibility of Δ_{n+1} follows straightforwardly from admissibility of Δ_n.
We observe that Δ_ω satisfies the following properties:

1. Δ_ω is decisive: for all $\varphi \in \mathcal{L}^+$, either $\varphi \in \Delta_\omega$ or $\neg\varphi \in \Delta_\omega$
2. Δ_ω is finitely $\vdash$-closed: if $\Delta' \subseteq \Delta_\omega$ finite and $\Delta' \vdash \varphi$, then $\varphi \in \Delta_\omega$
3. Δ_ω contains witnesses: if $@_{\mathsf{k}}\neg\Box\psi \in \Delta_\omega$, then there is a $\mathsf{j} \in \mathsf{J}$ with $@_{\mathsf{k}}\Diamond\mathsf{j}, @_{\mathsf{j}}\neg\psi \in \Delta_\omega$
4. Δ_ω is closed under all sequents $(@_{\mathsf{j}}\Theta \vdash @_{\mathsf{j}}\varphi) \in @\mathbf{AS}$: if $@_{\mathsf{j}}\Theta \subseteq \Delta_\omega$, then $@_{\mathsf{j}}\varphi \in \Delta_\omega$

Observe the weak formulation of property 2: closure under full derivability is not required. As a consequence, consistency of Δ_ω does not follow directly from properties 1 - 4: but it follows in an indirect manner, via the construction of the model M and the truth lemma (1) below.

Properties 1 and 2 are proved as usual. The witness property (3) is proved as follows. Suppose that $@_{\mathsf{k}}\neg\Box\psi \in \Delta_\omega$. Take n such that $\varphi_n = @_{\mathsf{k}}\neg\Box\psi$. We claim that $\Delta_n \vdash \varphi_n$, for otherwise $\neg\varphi_n \in \Delta_{n+1} \subseteq \Delta_\omega$ by the definition of Δ_{n+1}, while we have $\neg\varphi_n \notin \Delta_\omega$ by the decisiveness property of Δ_ω. Since $\Delta_n \vdash \varphi_n$, we have by the definition of Δ_{n+1} that, for some j, $@_{\mathsf{k}}\Diamond\mathsf{j}, @_{\mathsf{j}}\neg\psi \in \Delta_{n+1} \subseteq \Delta_\omega$.

For property 4, closure under $@\mathbf{AS}$, we argue as follows. Let $(\Theta \vdash \varphi) \in \mathbf{AS}$, $\mathsf{j} \in \mathsf{J}$, and suppose that $@_{\mathsf{j}}\Theta \subseteq \Delta_\omega$. By the definition of the collections M_ψ, we have that, for some $m \in M_{@_{\mathsf{j}}\varphi}$, $@_{\mathsf{j}}\Theta = \Theta_m$ and $@_{\mathsf{j}}\varphi = \varphi_m$. So $f_{@_{\mathsf{j}}\varphi}(m)$ is defined. Let $n = f_{@_{\mathsf{j}}\varphi}(m)$, then $n \in N_{@_{\mathsf{j}}\varphi}$, i.e. $\varphi_n = @_{\mathsf{j}}\varphi$. We claim that $\Delta_n \vdash \varphi_n$: this implies the desired result, for then $\varphi_n \in \Delta_{n+1} \subseteq \Delta_\omega$ by the definition of Δ_{n+1}. To prove the claim via contradiction, assume that $\Delta_n \nvdash \varphi_n$, then the first clause in the definition of Δ_{n+1} applies, for there is indeed an m with $f_{\varphi_n}(m) = n$. Now $\Delta_{n+1} = \Delta_n \cup \{\neg\varphi_n, \neg@_{\mathsf{j}}\theta\} \subseteq \Delta_\omega$ for some $\theta \in \Theta$; with $@_{\mathsf{j}}\Theta \subseteq \Delta_\omega$ we would have $\{@_{\mathsf{j}}\theta, \neg@_{\mathsf{j}}\theta\} \subseteq \Delta_\omega$, contradicting the decisiveness property of Δ_ω.

Now we know that Δ_ω satisfies the properties 1 to 4, we can define a model based on Δ_ω as follows. Let the relation $\sim$ on J be defined by

$$\mathsf{j} \sim \mathsf{k} \text{ iff } @_{\mathsf{j}}\mathsf{k} \in \Delta_\omega$$

Observe that, by the axioms $\mathbf{T}_@$, $\mathbf{B}_@$ and **Nom**, $\sim$ is an equivalence relation on J. We put

$$[\mathsf{j}] = \{\mathsf{k} \in \mathsf{J} \mid @_{\mathsf{j}}\mathsf{k} \in \Delta_\omega\}$$

so $[\mathsf{j}]$ is the $\sim$-equivalence class of j, and we have $[\mathsf{j}] = [\mathsf{k}] \Leftrightarrow \mathsf{j} \sim \mathsf{k}$ for all $\mathsf{j}, \mathsf{k} \in \mathsf{J}$. We construct the model $M = (W, R, V, A)$ from Δ_ω:

- $W = \{[\mathsf{j}] \mid \mathsf{j} \in \mathsf{J}\}(= \mathsf{J}/\!\sim)$,
- $R = \{([\mathsf{j}], [\mathsf{k}]) \mid @_{\mathsf{j}}\Diamond\mathsf{k} \in \Delta_\omega\}$,
- $V(p) = \{[\mathsf{j}] \mid @_{\mathsf{j}}p \in \Delta_\omega\}$
- $A(\mathsf{k}) = [\mathsf{k}]$

We claim that for all $\varphi \in \mathcal{L}^+$, all $\mathsf{j} \in \mathsf{J}$ and all substitutions $\sigma : \mathsf{J} \to \mathsf{J}$, the following Truth Lemma holds:

$$@_{\mathsf{j}}(\varphi\sigma) \in \Delta_\omega \Leftrightarrow (M_\sigma, [\mathsf{j}]) \models \varphi \tag{1}$$

This is proved with formula induction. The atomic cases are straightforward, and the propositional cases follow from the fact that Δ_ω is decisive and finitely $\vdash$-closed. For $\varphi = @_{\mathsf{k}}\psi$, (1) follows from $@_{\mathsf{j}}(@_{\mathsf{k}}\psi)\sigma \leftrightarrow @_{\sigma(\mathsf{k})}(\psi\sigma)$, an instance of **Agree**. For $\varphi = \Box\psi$, (1) comes down to

$$@_{\mathsf{j}}\Box\psi\sigma \in \Delta_\omega \Leftrightarrow \forall \mathsf{k} \in \mathsf{J}(@_{\mathsf{j}}\Diamond\mathsf{k} \in \Delta_\omega \Rightarrow (M_\sigma, [\mathsf{k}]) \models \psi)$$

Via contraposition, the induction hypothesis and the property $@_{\mathsf{j}}\chi \notin \Delta_\omega \Leftrightarrow @_{\mathsf{j}}\neg\chi \in \Delta_\omega$ (a consequence of decisiveness and finite $\vdash$-closedness), this follows from

$$@_{\mathsf{j}}\neg\Box\psi\sigma \in \Delta_\omega \Leftrightarrow \exists \mathsf{k} \in \mathsf{J}(@_{\mathsf{j}}\Diamond\mathsf{k} \in \Delta_\omega \ \&\ @_{\mathsf{k}}\neg\psi\sigma \in \Delta_\omega)$$

The $\Rightarrow$ part follows from the witness property of Δ_ω, the $\Leftarrow$ part from **Bridge** and the finite $\vdash$-closure property of Δ_ω.
Finally we consider (1) for the case $\varphi = \downarrow\!\mathsf{k}\psi$. This boils down to

$$@_{\mathsf{j}}(\downarrow\!\mathsf{k}'(\psi(\sigma[\mathsf{k} := \mathsf{k}']))) \in \Delta_\omega \Leftrightarrow @_{\mathsf{j}}(\psi(\sigma[\mathsf{k} := \mathsf{j}])) \in \Delta_\omega$$

where k' is fresh. This follows (via **Intr** and $\mathbf{K}_@$) from

$$\mathsf{j} \vdash \downarrow\!\mathsf{k}'(\psi(\sigma[\mathsf{k} := \mathsf{k}'])) \leftrightarrow \psi(\sigma[\mathsf{k} := \mathsf{j}])$$

which is a consequence of **DA** and the fact that $\psi(\sigma[\mathsf{k} := \mathsf{k}'])[\mathsf{k}' := \mathsf{j}] = \psi(\sigma[\mathsf{k} := \mathsf{j}])$ (since $\mathsf{k}' \notin \mathsf{nom}(\psi\sigma)$). This ends the proof of (1).

As a consequence of (1), we have (using that $\mathsf{i} \in \Delta_0 \subseteq \Delta_\omega$):

$$\varphi \in \Delta_\omega \Leftrightarrow (M, [\mathsf{i}]) \models \varphi$$

By the closure of Δ_ω with respect to @**AS**, we have for all $(\Theta, \varphi) \in \mathbf{AS}$ and all $\mathsf{j} \in \mathsf{J}$:

$$\text{if } (M, [\mathsf{j}]) \models \Theta, \text{ then } (M, [\mathsf{j}]) \models \varphi$$

So M is a named model satisfying **AS** with $(M, [\mathsf{i}]) \models \Gamma$. This ends the proof of the theorem. □

Although the model which is constructed in this proof satisfies Γ, it is not necessarily the case that the underlying *frame* satisfies the additional sequents. A modal logic is called *canonical* if all its axioms are valid on the frame underlying the canonical model. In the current context, where we have additional sequents rather than additional axioms that potentially characterize certain frame properties, we generalize this notion as follows. A hybrid logic is called canonical if all the sequents are valid on the frame underlying the canonical model. A sequent $\Gamma \vdash \varphi$ is *valid on a frame* $F = (W, R)$ iff for all models M based on F and every world $w \in W$, if $(M, w) \models \psi$ for all $\psi \in \Gamma$, then $(M, w) \models \varphi$. So, we do not show canonicity. However if we restrict the additional axiom sequents to those generated by pure sequents (where no propositional variables occur), we do get canonicity, due to the following lemma. Thus, for named models and pure formulas containing only finitely many nominals, truth in a model and validity in a frame coincide.

Lemma 4
Let $M = (F, V, A)$ be a named model and $\Gamma \vdash \varphi$ be a pure sequent. Suppose that for all pure instances $\Delta \vdash \psi$ of $\Gamma \vdash \varphi$, and for all $w \in W$, $(M, w) \models \Delta$ implies $(M, w) \models \psi$. Then $F \models \Gamma/\varphi$, i.e. for all V', A', w we have that $((F, V', A'), w) \models \Gamma$ implies $((F, V', A'), w) \models \varphi$. □

Proof (sketch) $A : \mathsf{I} \to W$ is surjective, so it has a right inverse $A^{-1} : W \to \mathsf{I}$ with $A(A^{-1}(w)) = w$. Define the nominal substitution $\sigma : \mathsf{I} \to \mathsf{I}$ by $\sigma(\mathsf{i}) = A^{-1}(A'(\mathsf{i}))$. Now we can prove, with straightforward formula induction, that for all pure formulas θ:

$$((F, V, A), w) \models \theta\sigma \Leftrightarrow ((F, V', A'), w) \models \theta$$

This implies the lemma. □

5 Application to non-compact modal logics

We provide a number of interesting instances of Theorem 2. These are examples of cases where pure axioms do not suffice to obtain completeness for the relevant class of models, but pure sequents do. Since pure axioms are a special case of pure sequents, these are generalizations of the known result for hybrid logic [6]. Thus, if **AS** contains only pure formulas and finitely many nominals, not only the model provided by Theorem 2, but also the frame underlying it, validates **AS**. In this section by Khyb_ω + **AS** we mean Khyb_ω extended with all pure instances of **AS**.

5.1 Hybrid ancestral logic

Ancestral logic is the modal logic with two modalities $\Box$ and $\Box^*$, where the accessibility relation associated with the latter is equal to or a subset of the reflexive transitive closure of the accessibility relation associated with the former. Ancestral logic is non-compact, and in fact a counterexample to compactness gives the inspiration for a suitable hybrid version of this logic, namely Khyb_ω extended with a countable set of pure sequents containing only finitely many nominals. Let $\Box^n\varphi$ stand for φ preceded by n $\Box$-operators.

AS1 $\{@_{\mathsf{i}}\Box^n\neg\mathsf{j} \mid n \in \mathbb{N}\} \vdash @_{\mathsf{i}}\Box^*\neg\mathsf{j}$

It is clear that **AS1** is valid exactly in those frames in which the accessibility relation of $\Box^*$ is the reflexive transitive closure for the accessibility relation of $\Box$. Thus, by Theorem 2 and Lemma 4, Khyb_ω + **AS1** is strongly complete with respect to such frames.

5.2 Hybrid reachability logic

Let us define Hybrid reachability logic as the hybrid logic given by Khyb_ω + **AS2**, as follows, where $\Diamond^n\varphi$ stands for φ preceded by n $\Diamond$-operators:

AS2 $\{\neg@_{\mathsf{i}}\Diamond^n\mathsf{j} \mid n \in \mathbb{N}\} \vdash \bot$

It is clear that **AS2** is valid exactly in those frames in which the accessibility relation R is reachable, in the sense that for any two states i, j in the model either $i = j$ or there is a sequence $s_0R\ldots s_n$ where $s_0 = i$ and $s_n = j$, where $n \geq 1$. Thus, by Theorem 2 and Lemma 4, Khyb_ω + **AS2** is strongly complete with respect to reachable frames.

5.3 Hybrid cycle logic

Let us define Hybrid cycle logic as the hybrid logic given by Khyb_ω + **AS3**, as follows:

AS3 $\{\neg@_{\mathsf{i}}\Diamond^n\mathsf{i} \mid n \in \mathbb{N}, n \geq 1\} \vdash \bot$

It is clear that **AS3** is valid exactly in those frames in which the accessibility relation R contains cycles from every world, in the sense that for any state i in the model, there is some sequence $s_0 R \ldots s_n$ where $s_0 = i$ and $s_n = i$, where $n \geq 1$. Thus, by Theorem 2 and Lemma 4, Khyb_ω + **AS3** is strongly complete with respect to frames with cycles.

Note that the set of pure axioms $\{\neg @_\mathsf{i} \Diamond^n \mathsf{i} \mid n \in \mathbb{N}, n \geq 1\}$ characterizes the class of frames in which there are no cycles from any world. However, for the corresponding completeness result, the finitary methods of [6] suffice.

5.4 Hybrid BCC logic

BCC-logic is the logic of the bounded chain condition, as defined in [16]: for all states i, there is a bound $n \in \mathbb{N}$ such that for any j there are only chains $iR \ldots j$ of length smaller than n from i to j.

Let us define Hybrid BCC logic as the hybrid logic given by Khyb_ω + **AS4**, the infinitary rule of [16], which turns out to be pure and does not contain any nominals:

AS4 $\quad \{\Diamond^n \top \mid n \in \mathbb{N}\} \vdash \perp$

Thus, by Theorem 2 and Lemma 4, Khyb_ω + **AS4** is strongly complete with respect to frames with the bounded chain condition.

Note that the BCC-condition is stronger than converse wellfoundedness (no infinite ascending chains), for which there is no characterizing countable set of sequents containing only finitely many nominals. This follows from results on the undefinability of well-orderings in infinitary first order predicate logic by Lopez-Escobar [11] and the embedding of hybrid logic into first-order predicate logic.

6 Relation to the literature

This paper brings together two research areas. The first is investigations into strong completeness for non-compact logics. The other is general completeness proofs for hybrid logic extended with arbitrary pure axioms.

When non-compactness is caused by a modality that is interpreted as a (reflexive) transitive closure of another modality, it seems rather natural to consider infinitary proof systems to deal with the infinitary character of these modal operators. Many logics with such a modality can be seen as fragment of propositional dynamic logic (PDL, see [8] for a textbook introduction).

Goldblatt's [7], Segerberg's [16] and our [14] contain general results on strong completeness of infinitary modal logics like PDL. It is shown that adding denumerably many rules that satisfy certain properties to a basic proof system, yields a proof system that is strongly complete with respect to the appropriate class of models.

Segerberg's [16], too, is devoted to proving strong completeness for a class of noncompact modal logics. Several examples are discussed: most of them are logics with (reflexive) transitive closure modalities. Interestingly, it also includes logics that are not compact due to other reasons, e.g. logics that satisfy the *bounded chain condition* (see Section 5). All these logics are axiomatized by adding infinitary rules to a finitary proof system for modal logic. Like Goldblatt, Segerberg considers a very general case, where the only requirement is that the set of all instances of the infinitary rules is countable (all the examples mentioned above fall into this category).

The completeness result for hybrid logic extended with arbitrary pure axioms was first shown in [6], and is also presented in [3], which provided the basis for the proof in the textbook [2], by which we learned about the result. More results based on this result and references can be found in [17].

This paper brings together these areas of non-compact and hybrid logics, but there are earlier results. In [12], Passy and Tinchev investigate hybrid versions of PDL and present an infinitary proof system for the language with nominals and the universal modality, which is shown to be strongly complete. In [10], we presented an infinitary proof system for hybrid logic without the nominal binder $\downarrow$ that could be extended with countably many pure sequents. This paper elaborates and generalizes the results in that paper.

Via the standard translation of modal logic to first-order logic, the hybrid logic with @ and $\downarrow$ corresponds with the bounded fragment of first-order logic [18]. This fragment was investigated by Feferman in [4], but in a different infinitary setting: he considers countable conjunctions and disjunctions, but finite sequents. This is the opposite of what we do: finite formulas and countable sequents. Since countable sequents can be reduced to finite sequents in the presence of countable conjunctions and disjunctions, our results are expressible in the bounded fragment considered by Feferman. It may be interesting to study this embedding more closely.

It is also interesting to investigate the frame properties corresponding to some typical non-compact modal logics, such as those treated in Section 5. It turns out that the properties corresponding to hybrid ancestral logic (the accessibility relation associated with $\Box^*$ is included in the reflexive transitive closure of the accessibility relation associated with $\Box$), hybrid reachability logic (for any two states, one can get from one to the other in finitely many but arbitrarily many steps), and hybrid cycle logic (every state is part of a finite, but arbitrarily long, cycle of steps), can all be expressed in first-order logic with a transitive closure operator. Thus, they do not need full second-order logic. An interesting treatment of the expressivity of first-order logic plus transitive closure (possibly with a pairing function) can be found in [1]. This correspondence suggests an alternative approach to axiomatising the logics of Section 5, namely by extending the language of hybrid logic with some transitive-closure-like operators, and then trying to characterize interesting frame properties using the infinitary base system Khyb_ω plus a single axiom expressing the relevant property, and then applying an adapted version of [6]. However, it is far from obvious whether such an approach would work, as correspondence theory generally does not deliver results in such a straightforward way (see also [18, 2]).

7 Conclusion and further research

In this paper we provided a strongly complete infinitary proof system for hybrid logic. The completeness proof worked in such a way that we immediately derived completeness for logics that extend the proof system with countably many axiom sequents. This allowed us to obtain strongly complete proof systems for non-compact hybrid logics. If the additional axiom sequents are pure and contain only finitely many nominals, then we automatically have canonicity in the sense that on the canonical frames of these logics, all their sequents are valid [2].

Extending the results presented here to other versions of hybrid logic (e.g. with a universal modality added, or without the nominal binder $\downarrow$) is straightforward. When $\downarrow$ is not present, the Name rule **NR** and the Paste rule **PR** are not derivable anymore, and have to be added as proof rules.

In Chapter 4 of [17], Ten Cate answers the question which elementary frame classes are definable by a set of pure *formulas* of hybrid logic. It would be interesting to investigate the analogous question for the sets of pure *sequents* we use in this paper, for example, building on the relation of our logics to infinitary versions of the bounded fragment of first-order logic [4]. Moreover, it would be interesting to investigate the complexity of the derivability relation for infinitary hybrid logic plus various classes of well-behaved axiom sequents. For example, it would be interesting to investigate the complexity of the validity problem for infinitary hybrid logics restricted to rules with recursively enumerable antecedents (such as all example logics of Section 5). In the future we hope to attain similar results for hybrid logic with uncountably many pure rules with countably many nominals. This would yield strongly complete proof systems for many other interesting classes of frames.

8 Acknowledgments

We would like to thank Balder ten Cate, Benedikt Löwe, Yde Venema, and two anonymous referees for their fruitful suggestions.

References

[1] A. Avron. Transitive closure and the mechanization of mathematics. In F. Kamareddine, editor, *Thirty-five Years of Automating Mathematics*, pages 149 – 171. Kluwer Academic Publishers, Dordrecht (the Netherlands), 2003.

[2] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 2001.

[3] P. Blackburn and M. Tzakova. Hybrid languages and temporal logic. *Logic Journal of the IGPL*, 7(1):27–54, 1999.

[4] S. Feferman. Persistent and invariant formulas for outer extensions. *Compositio Mathematica*, 20:29–52, 1969.

[5] M. J. Gabbay and A. M. Pitts. A new approach to abstract syntax with variable binding. *Formal Aspects of Computing*, 13(3-5):341–363, 2002.

[6] G. Gargov, S. Passy, and T. Tinchev. Modal environment for Boolean speculations. In D. Skordev, editor, *Mathematical logic and its Applications. Proceedings of the Summer School and Conference dedicated to the 80th Anniversary of Kurt Gödel (Druzhba, 1986)*, pages 253–263. Plenum Press, 1987.

[7] R. Goldblatt. *Mathematics of Modality*, volume 43 of *CSLI Lecture Notes*. CSLI Publications, Stanford, California, 1993.

[8] D. Harel, D. Kozen, and J. Tiuryn. *Dynamic Logic*. Foundation of Computing. MIT Press, Cambrigde, Massachusetts, 2000.

[9] H. J. Keisler. *Model Theory for Infinitary Logic: Logic with Countable Conjunctions and Finite Quantifiers*. North-Holland Publishing Company, Amsterdam, 1971.

[10] B. P. Kooi, G. R. Renardel de Lavalette, and R. Verbrugge. Strong completeness for non-compact hybrid logics. In Renate Schmidt, Ian Pratt-Hartmann, and Mark Reynolds, editors, *AiML2004 — Advances in Modal Logic (conference proceedings)*, pages 212–223. Department of Computer Science, University of Manchester, Technical Report Series UMCS-04-9-1, 2004.

[11] E. G. K. Lopez-Escobar. On defining well-orderings. *Fundamenta Mathematicae*, 59:13–21,299–300, 1966.

[12] S. Passy and T. Tinchev. An essay in combinatory dynamic logic. *Information and Computation*, 93:263–332, 1991.

[13] A. N. Prior. *Past, Present and Future*. Oxford University Press, 1967.

[14] G. R. Renardel de Lavalette, B. P. Kooi, and R. Verbrugge. Strong completeness for propositional dynamic logic. In Philippe Balbiani, Nobu-Yuki Suzuki, and Frank Wolter, editors, *AiML2002 — Advances in Modal Logic (conference proceedings)*, pages 377–393. Institut de Recherche en Informatique de Toulouse IRIT, 2002.

[15] G. R. Renardel de Lavalette, B. P. Kooi, and R. Verbrugge. Strong completeness and limited canonicity for PDL. Submitted, 2005.

[16] K. Segerberg. A model existence theorem in infinitary propositional modal logic. *Journal of Philosophical Logic*, 23:337–367, 1994.

[17] B. D. ten Cate. *Model theory for extended modal languages*. ILLC Dissertation Series DS-2005-01, Amsterdam, 2005. Available online at `http://www.illc.uva.nl/Publications/`.

[18] J.F.A.K. van Benthem. *Modal Logic and Classical Logic*. Bibliopolis, Naples, 1983.