Correct Grounded Reasoning with Presumptive Arguments

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Abstract. We address the semantics and normative questions for reasoning with presumptive arguments: How are presumptive arguments grounded in interpretations; and when are they evaluated as correct? For deductive and uncertain reasoning, classical logic and probability theory provide canonical answers to these questions. Staying formally close to these, we propose case models and their preferences as formal semantics for the interpretation of presumptive arguments. Arguments are evaluated as presumptively valid when they make a case that is maximally preferred. By qualitative and quantitative representation results, we show formal relations between deductive, uncertain and presumptive reasoning. In this way, the work is a step to the connection of logical and probabilistic approaches in AI.

1 Introduction

There is a growing and productive research community in artificial intelligence focusing on argumentation. Some use artificial intelligence tools to study natural argumentation, others focus on computational properties, and there is work on formal foundations [12]. The present paper considers the formal foundations of argumentation, interpreted as reasoning with presumptive, possibly defeasible arguments. Studying the formal foundations of argumentation can help answering two questions:

1. **The semantics question.** How are presumptive arguments grounded in interpretations? This question is about *grounded argumentation*.
2. **The normative question.** When are presumptive arguments evaluated as correct? This question is about *correct argumentation*.

For deductive and uncertain reasoning, canonical answers to these questions exist. For deductive reasoning, arguments are interpreted in logical models (question 1), and logical validity characterizes correct deductive reasoning, such as formal proof (question 2). Uncertain reasoning is interpreted in probability distributions, and the probability calculus characterizes correct uncertain reasoning, such as Bayesian updating.

For reasoning with presumptive arguments, the answers to the two questions are less well-developed. In today's state of the art, a key role is played by Dung's groundbreaking work on abstract argumentation [11]. One can say that Dung's work provides an answer to the semantics question 1 by interpreting argument attack in directed graphs, and to the normative question 2 by formalizing an argumentative winning criterion in terms of argument admissibility. We distinguish two complications.

One complication is that these answers depend on the choice of one of the available abstract argumentation semantics. Dung himself suggested several extension types as interpretations of directed graphs (the grounded, complete, preferred, stable extensions; each based on the important notion of admissible set), and the number of proposals expanded quickly thereafter (see the review [3]).

A second complication is that these answers focus on argument attack, abstracting from argument support. Extending to include argument support has led to a variety of approaches, some referred to as structured argumentation. A recent special issue of the journal 'Argument and Computation' [5] usefully explains how four leading models connect in different ways to Dung's semantics: ABA [8], ASPIC+ [22], DeLP [13], deductive argumentation [6].

We propose case models and their preferences as a formal semantics used for the interpretation of presumptive arguments (answering question 1), and evaluate arguments as acceptable when they make a case that is maximally preferred (answering question 2). The proposed formalism is designed in close connection with classical logic and probability theory, in order to show formal relations between deductive, uncertain and presumptive reasoning. The formalism presented builds on an existing line of research [29–32]. [29, 30] study formal connections between arguments, logic and probabilities, but do not provide a model-theoretic semantics as we do here. The case model semantics presented here formalizes ideas semi-formally presented in [31], which in turn is inspired by issues arising when modeling argument-based and scenario-based evidential reasoning about crimes using Bayesian networks [32] (cf. the discussion in Section 4).

2 General idea

The argumentation theory developed in this paper considers arguments that can be presumptive (also called ampliative), in the sense of logically going beyond their premises. Against the background of classical logic, an argument from premises P to conclusions Q goes beyond its premises when Q is not logically implied by P. Many arguments used in practice are presumptive. For instance, the prosecution may argue that a suspect was at the crime scene on the basis of a witness testimony. The fact that the witness has testified as such does not logically imply the fact that the suspect was at the crime scene. In particular, when the witness testimony is intentionally false, based on inaccurate observations or inaccurately remembered, the suspect may not have been at the crime scene at all. Denoting the witness testimony by P and the suspect being at the crime scene as Q, the argument from P to Q is presumptive since P does not logically imply Q. For presumptive arguments, it is helpful to consider the *case made by the argument*, defined as the conjunction of the premises and conclusions of the argument [28, 29]. The case made by the argument from P to Q is $P \wedge Q$, using the conjunction of classical logic. An example of a non-presumptive argument goes from $P \wedge Q$ to Q. Here Q is logically implied by $P \wedge Q$. Presumptive arguments are often defeasible [20, 27], in the sense that extending the premises may lead to the retraction of conclusions.

Figure 1 shows (on the left) two presumptive arguments from the same premises P. They make cases that are conflicting: one supports the case $P \wedge Q$, the other the

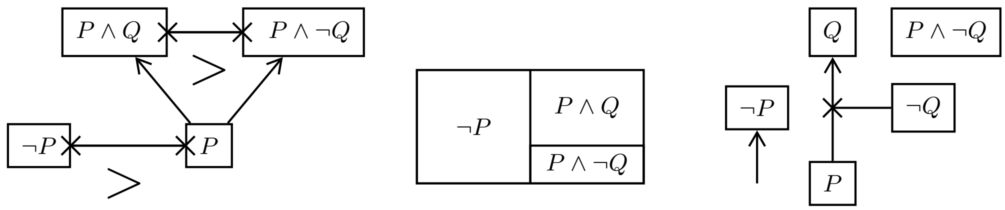

Fig. 1. Arguments and cases

case $P \wedge \neg Q$. The $>$-sign indicates that one argument makes a stronger case than the other, resolving the conflict: the argument for the case $P \wedge Q$ is stronger than that for $P \wedge \neg Q$. The figure also shows two assumptions P and $\neg P$, that can be considered as arguments from logically tautologous premises. Here the assumption $\neg P$ makes the strongest case when compared to the assumption P. Logically such assumptions can be treated as arguments from logical truth $\top$. In the figure on the right one sees which conclusions follow presumptively from which premises: $\neg P$ follows as an assumption, and Q follows from P. $\neg Q$ blocks the inference from P to Q. From premises $P \wedge \neg Q$ no further conclusions follow. The arguments make three cases: $\neg P$, $P \wedge Q$ and $P \wedge \neg Q$ (Figure 1; middle). Their sizes suggest a preference relation.

The comparison of arguments and of cases are closely related in our approach, which can be illustrated as follows. The idea is that a case is preferred to another case if there is an argument with premises that supports the former case more strongly than the latter case. Hence, in the example in the figures, $\neg P$ is preferred to both $P \wedge Q$ and $P \wedge \neg Q$, and $P \wedge Q$ is preferred to $P \wedge \neg Q$. Conversely, given the cases and their preferences, we can compare arguments. The argument from P to Q is stronger than from P to Q' when the best case that can be made from $P \wedge Q$ is preferred to the best case that can be made from $P \wedge Q'$.

3 Formalism and properties

We now formalize case models and how they can be used to interpret arguments (Section 3.1). Then follow qualitative and quantitative representation results (Sections 3.2 and 3.3).

3.1 Case models and arguments

The formalism uses a classical logical language L generated from a set of propositional constants in a standard way. We write $\neg$ for negation, $\wedge$ for conjunction, $\vee$ for disjunction, $\leftrightarrow$ for equivalence, $\top$ for a tautology, and $\bot$ for a contradiction. The associated classical, deductive, monotonic consequence relation is denoted $\models$. We assume a finitely generated language.

First we define case models, formalizing the idea of cases and their preferences. The cases in a case model must be logically consistent, mutually incompatible and different; and the comparison relation must be total and transitive (hence is what is called a total preorder, commonly modeling preference relations [23]).

Definition 1 *A* case model *is a pair* $(C, \geq)$ *with finite* $C \subseteq L$*, such that the following hold, for all* φ*,* ψ *and* $\chi \in C$*:*

1. $\not\models \neg\varphi$*;*
2. *If* $\not\models \varphi \leftrightarrow \psi$*, then* $\models \neg(\varphi \wedge \psi)$*;*
3. *If* $\models \varphi \leftrightarrow \psi$*, then* $\varphi = \psi$*;*
4. $\varphi \geq \psi$ *or* $\psi \geq \varphi$ *;*
5. *If* $\varphi \geq \psi$ *and* $\psi \geq \chi$*, then* $\varphi \geq \chi$*.*

The strict weak order $>$ standardly associated with a total preorder $\geq$ is defined as $\varphi > \psi$ if and only if it is not the case that $\psi \geq \varphi$ (for φ and $\psi \in C$). When $\varphi > \psi$, we say that φ is (strictly) preferred to ψ. The associated equivalence relation $\sim$ is defined as $\varphi \sim \psi$ if and only if $\varphi \geq \psi$ and $\psi \geq \varphi$.

Example. Figure 1 shows a case model with cases $\neg P$, $P \wedge Q$ and $P \wedge \neg Q$. $\neg P$ is (strictly) preferred to $P \wedge Q$, which in turn is preferred to $P \wedge \neg Q$.

Next we define arguments from premises $\varphi \in L$ to conclusions $\psi \in L$.

Definition 2 *An* argument *is a pair* (φ, ψ) *with* φ *and* $\psi \in L$*. The sentence* φ *expresses the argument's premises, the sentence* ψ *its conclusions, and the sentence* $\varphi \wedge \psi$ *the* case made *by the argument. Generalizing, a sentence* $\chi \in L$ *is a* premise *of the argument when* $\varphi \models \chi$*, a* conclusion *when* $\psi \models \chi$*, and a* position *in the case made by the argument when* $\varphi \wedge \psi \models \chi$*. An argument* (φ, ψ) *is* (properly) presumptive *when* $\varphi \not\models \psi$*; otherwise* non-presumptive*. An argument* (φ, ψ) *is an* assumption *when* $\models \varphi$*, i.e., when its premises are logically tautologous.*

Note our use of the plural for an argument's premises, conclusions and positions. This terminological convention allows us to speak of the premises $\mathtt{p}$ and $\neg\mathtt{q}$ and conclusions $\mathtt{r}$ and $\neg\mathtt{s}$ of the argument $(\mathtt{p} \wedge \neg\mathtt{q}, \mathtt{r} \wedge \neg\mathtt{s})$. Also the convention fits our non-syntactic definitions, where for instance an argument with premise χ also has logically equivalent sentences such as $\neg\neg\chi$ as a premise.

Coherent arguments are defined as arguments that make a case that is logically implied by a case in the case model.

Definition 3 *Let* $(C, \geq)$ *be a case model. Then we define, for all* φ *and* $\psi \in L$*:*

$(C, \geq) \models (\varphi, \psi)$ *if and only if* $\exists \omega \in C$*:* $\omega \models \varphi \wedge \psi$*.*

We then say that the argument from φ *to* ψ *is* coherent *with respect to the case model. We define, for all* φ *and* $\psi \in L$*:*

$(C, \geq) \models \varphi \Rightarrow \psi$ *if and only if* $\exists \omega \in C$*:* $\omega \models \varphi \wedge \psi$ *and* $\forall \omega \in C$*: if* $\omega \models \varphi$*, then* $\omega \models \varphi \wedge \psi$*.*

We then say that the argument from φ *to* ψ *is* conclusive *with respect to the case model.*

Example (continued). In the case model of Figure 1, the arguments from $\top$ to $\neg P$ and to P, and from P to Q and to $\neg Q$ are coherent and not conclusive in the sense of this definition. Denoting the case model as $(C, \geq)$, we have $(C, \geq) \models (\top, \neg P)$, $(C, \geq) \models (\top, P)$, $(C, \geq) \models (P, Q)$ and $(C, \geq) \models (P, \neg Q)$. The arguments from a case (in the case model) to itself, such as from $\neg P$ to $\neg P$, or from $P \wedge Q$ to $P \wedge Q$ are conclusive. The argument $(P \vee R, P)$ is also conclusive in this case model, since all $P \vee R$-cases are P-cases. Similarly, $(P \vee R, P \vee S)$ is conclusive.

The notion of presumptive validity considered here is based on the idea that some arguments make a better case than other arguments from the same premises. More precisely, an argument is presumptively valid if there is a case implying the case made by the argument that is at least as preferred as all cases implying the premises.

Definition 4 *Let $(C, \geq)$ be a case model. Then we define, for all φ and $\psi \in L$:*

$(C, \geq) \models \varphi \rightsquigarrow \psi$ *if and only if* $\exists \omega \in C$:
1. $\omega \models \varphi \wedge \psi$*; and*
2. $\forall \omega' \in C$: *if* $\omega' \models \varphi$*, then* $\omega \geq \omega'$.

We then say that the argument from φ to ψ is (presumptively) valid *with respect to the case model. A presumptively valid argument is* defeasible*, when it is not conclusive.*

Circumstances χ are defeating *when $(\varphi \wedge \chi, \psi)$ is not presumptively valid. Defeating circumstances are* rebutting *when $(\varphi \wedge \chi, \neg\psi)$ is presumptively valid; otherwise they are* undercutting. *Defeating circumstances are* excluding *when $(\varphi \wedge \chi, \psi)$ is not coherent.*

Example (continued). In the case model of Figure 1, the arguments from $\top$ to $\neg P$, and from P to Q are presumptively valid in the sense of this definition. Denoting the case model as $(C, \geq)$, we have formally that $(C, \geq) \models \top \rightsquigarrow \neg P$ and $(C, \geq) \models P \rightsquigarrow Q$. The coherent arguments from $\top$ to P and from P to $\neg Q$ are not presumptively valid in this sense.
Example. Arguments typically consist of multiple steps. Figure 2 shows a two step argument on the left. The first step is from P to Q, the second from Q to R. Both steps have defeating circumstances: the first $\neg Q$, the second $\neg R$. In the case model shown, Q follows presumptively from P since $P \wedge Q \wedge R$ is a (here: the) preferred case given P. From Q follows R. $\neg Q$ is defeating for the former presumptive inference, since there is no preferred case of $P \wedge \neg Q$ in which R holds. (That preferred case is $P \wedge \neg Q$.) $\neg R$ is a defeater for the second presumptive inference. Formally, we have:

$$
\begin{array}{ll}
(C, \geq) \models P \rightsquigarrow Q & \qquad (C, \geq) \not\models P \wedge \neg Q \rightsquigarrow Q \\
(C, \geq) \models Q \rightsquigarrow R & \qquad (C, \geq) \not\models Q \wedge \neg R \rightsquigarrow R
\end{array}
$$

Note that in the case model also the following hold:

$$
\begin{array}{ll}
(C, \geq) \models P \rightsquigarrow Q \wedge R & \qquad (C, \geq) \models Q \Rightarrow P \\
(C, \geq) \models R \Rightarrow P \wedge Q & \qquad (C, \geq) \models R \Rightarrow Q \\
(C, \geq) \models \top \rightsquigarrow \neg P &
\end{array}
$$

The following properties follow directly from the definitions. Conclusive arguments are coherent, but there are case models with a coherent, yet inconclusive argument. Conclusive arguments are presumptively valid, but there are case models with a presumptively valid, yet inconclusive argument. Presumptively valid arguments are coherent, but there are case models with a coherent, yet presumptively invalid argument. The next proposition provides key logical properties of this notion of presumptive validity.

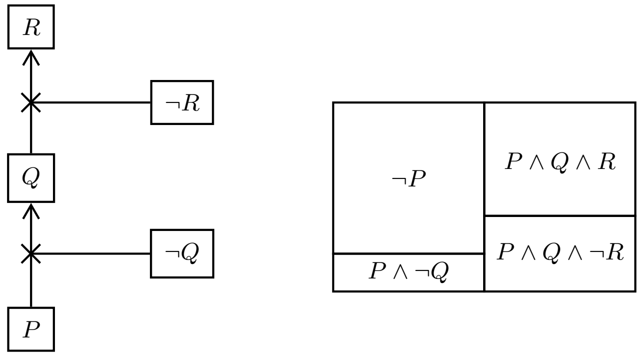

Fig. 2. An argument with two steps, each with exceptions

Many have been studied for nonmonotonic inference relations [4, 16, 18]. Given a case model $(C, \geq)$, we write $\varphi \mid\!\sim \psi$ for $(C, \geq) \models \varphi \rightsquigarrow \psi$. We write $C(\varphi)$ for the set $\{\omega \in C \mid \omega \models \varphi\}$.

(LE), for Logical Equivalence, expresses that in a valid argument the premises and the conclusions can be replaced by a logical equivalent (in the sense of $\models$). (Cons), for Consistency, expresses that the conclusions of presumptively valid arguments must be logically consistent. (Ant), for Antededence, expresses that when certain premises validly imply a conclusion, the case made by the argument is also validly implied by these premises. (RW), for Right Weakening, expresses that when the premises validly imply a composite conclusion also the intermediate conclusions are validly implied. (CCM), for Conjunctive Cautious Monotony, expresses that the case made by a valid argument is still validly implied when an intermediate conclusion is added to the argument's premises. (CCT), for Conjunctive Cumulative Transitivity, is a variation of the related property Cumulative Transitivity property (CT, also known as Cut). (CT)—extensively studied in the literature—has $\varphi \mid\!\sim \chi$ instead of $\varphi \mid\!\sim \psi \wedge \chi$ as a consequent. The variation is essential in our setting where the (And) property does not hold generally (If $\varphi \mid\!\sim \psi$ and $\varphi \mid\!\sim \chi$, then $\varphi \mid\!\sim \psi \wedge \chi$). Assuming (Ant), (CCT) expresses the validity of chaining valid implication from φ via the case made in the first step $\varphi \wedge \psi$ to the case made in the second step $\varphi \wedge \psi \wedge \chi$. (See [28, 29], introducing (CCT).)

Proposition 5 *Let $(C, \geq)$ be a case model. For all φ, ψ and $\chi \in L$:*

(LE) *If $\varphi \mid\!\sim \psi$, $\models \varphi \leftrightarrow \varphi'$ and $\models \psi \leftrightarrow \psi'$, then $\varphi' \mid\!\sim \psi'$.*
(Cons) $\varphi \not\mid\!\sim \bot$.
(Ant) *If $\varphi \mid\!\sim \psi$, then $\varphi \mid\!\sim \varphi \wedge \psi$.*
(RW) *If $\varphi \mid\!\sim \psi \wedge \chi$, then $\varphi \mid\!\sim \psi$.*
(CCM) *If $\varphi \mid\!\sim \psi \wedge \chi$, then $\varphi \wedge \psi \mid\!\sim \chi$.*
(CCT) *If $\varphi \mid\!\sim \psi$ and $\varphi \wedge \psi \mid\!\sim \chi$, then $\varphi \mid\!\sim \psi \wedge \chi$.*

Proof. (LE): Direct from the definition. (Cons): Otherwise there would be an inconsistent element of C, contradicting the definition of a case model. (Ant): When $\varphi \mid\!\sim \psi$, there is an ω with $\omega \models \varphi \wedge \psi$ that is $\geq$-maximal in $C(\varphi)$. Then also $\omega \models \varphi \wedge \varphi \wedge \psi$, hence $\varphi \mathrel{|\!\sim} \varphi \wedge \psi$. (RW): When $\varphi \mathrel{|\!\sim} \psi \wedge \chi$, there is an $\omega \in C$ with $\omega \models \varphi \wedge \psi \wedge \chi$ that is maximal in $C(\varphi)$. Since then also $\omega \models \varphi \wedge \psi$, we find $\varphi \mathrel{|\!\sim} \psi$. (CCM): By the assumption, we have an $\omega \in C$ with $\omega \models \varphi \wedge \psi \wedge \chi$ that is maximal in $C(\varphi)$. Since $C(\varphi \wedge \psi) \subseteq C(\varphi)$, ω is also maximal in $C(\varphi \wedge \psi)$, and we find $\varphi \wedge \psi \mathrel{|\!\sim} \chi$. (CCT): Assuming $\varphi \mathrel{|\!\sim} \psi$, there is an $\omega \in C$ with $\omega \models \varphi \wedge \psi$, maximal in $C(\varphi)$. Assuming also $\varphi \wedge \psi \mathrel{|\!\sim} \chi$, there is an $\omega' \in C$ with $\omega \models \varphi \wedge \psi \wedge \chi$, maximal in $C(\varphi \wedge \psi)$. Since $\omega \in C(\varphi \wedge \psi)$, we find $\omega' \geq \omega$. By transitivity of $\geq$, and the maximality of ω in $C(\varphi)$, we therefore have that ω' is maximal in $C(\varphi)$. As a result, $\varphi \mathrel{|\!\sim} \psi \wedge \chi$. QED

We speak of *coherent premises* when the argument from the premises to themselves is coherent. The following proposition provides some equivalent characterizations of coherent premises.

Proposition 6 *Let $(C, \geq)$ be a case model. The following are equivalent, for all $\varphi \in L$:*

1. $\varphi \mathrel{|\!\sim} \varphi$;
2. $\exists \omega \in C : \omega \models \varphi$ *and* $\forall \omega' \in C$: *If* $\omega' \models \varphi$, *then* $\omega \geq \omega'$;
3. $\exists \omega \in C : \varphi \mathrel{|\!\sim} \omega$.
4. $\exists \omega \in C : \omega \models \varphi$.

Proof. 1 and 2 are equivalent by the definition of $\mathrel{|\!\sim}$. Assume 2. Then there is a $\geq$-maximal element ω of $C(\varphi)$. By the definition of $\mathrel{|\!\sim}$, then $\varphi \mathrel{|\!\sim} \omega$; proving 3. Assume 3. Then there is a $\geq$-maximal element ω' of $C(\varphi)$ with $\omega' \models \varphi \wedge \omega$. For this ω' also $\omega' \models \varphi$, showing 2. 4 logically follows from 2. Assume 4. Hence $C(\varphi)$ is a finite, non-empty set, and 2 follows. QED

Hence coherent and presumptively valid arguments have coherent premises. The next corollary shows that logical generalisations of coherent premises are coherent.

Corollary 7 *Let $(C, \geq)$ be a case model. Then:*

If $\varphi \mathrel{|\!\sim} \varphi$ and $\varphi \models \psi$, then $\psi \mathrel{|\!\sim} \psi$.

In the representation result of the next subsection, additional properties are needed. We use the set of case expressions $L^* \subseteq L$ consisting of the logical combinations of the cases of the case model using negation, conjunction and logical equivalence (cf. the algebra underlying probability functions [23]).

(Coh), for Coherence, expresses that coherent premises correspond to a consistent case expression implying the premises. (Ch), for Choice, expresses that, given two coherent case expressions, at least one of three options follows validly: the conjunction of the case expression, or the conjunction of one of them with the negation of the other. (OC), for Ordered Choice, expresses that preferred choices between case expressions are transitive. Here we say that a case expression is a *preferred choice* over another, when the former follows validly from the disjunction of both. A *preferred case* given certain premises is a case that presumptively follows from those premises.

Proposition 8 *Let $(C, \geq)$ be a case model, and $L^* \subseteq L$ the closure of C under negation, conjunction and logical equivalence. Writing $\mathrel{|\!\sim}^*$ for the restriction of $\mathrel{|\!\sim}$ to L^*, we have, for all φ, ψ and $\chi \in L^*$:*

(Coh) $\varphi \mathrel{|\!\sim} \varphi$ *if and only if* $\exists\varphi^* \in L^*$ *with* $\varphi^* \not\models \bot$ *and* $\varphi^* \models \varphi$*;*

(Ch) *If* $\varphi \mathrel{|\!\sim}^* \varphi$ *and* $\psi \mathrel{|\!\sim}^* \psi$*, then* $\varphi \vee \psi \mathrel{|\!\sim}^* \neg\varphi \wedge \psi$ *or* $\varphi \vee \psi \mathrel{|\!\sim}^* \varphi \wedge \psi$ *or* $\varphi \vee \psi \mathrel{|\!\sim}^* \varphi \wedge \neg\psi$*;*

(OC) *If* $\varphi \vee \psi \mathrel{|\!\sim}^* \varphi$ *and* $\psi \vee \chi \mathrel{|\!\sim}^* \psi$*, then* $\varphi \vee \chi \mathrel{|\!\sim}^* \varphi$*.*

Proof. (Coh): By Proposition 6, $\varphi \mathrel{|\!\sim} \varphi$ if and only if there is an $\omega \in C$ with $\omega \models \varphi$. The property (Coh) follows since $C \subseteq L^*$ and, for all consistent $\varphi^* \in L^*$, there is an $\omega \in C$ with $\omega \models \varphi^*$.
(Ch): Consider sentences φ and $\psi \in L^*$ with $\varphi \mathrel{|\!\sim}^* \varphi$ and $\psi \mathrel{|\!\sim}^* \psi$. Then, by Corollary 7, $\varphi \vee \psi \mathrel{|\!\sim} \varphi \vee \psi$. By Proposition 6, there is an $\omega \in C$, with $\omega \models \varphi \vee \psi$. The sentences φ and ψ are elements of L^*, hence also the sentences $\varphi \wedge \neg\psi$, $\varphi \wedge \psi$ and $\neg\varphi \wedge \psi \in L^*$. All are logically equivalent to disjunctions of elements of C (possibly the empty disjunction, logically equivalent to $\bot$). Since $\omega \models \varphi \vee \psi$, $\models \varphi \vee \psi \leftrightarrow (\varphi \wedge \neg\psi) \vee (\varphi \wedge \psi) \vee (\neg\varphi \wedge \psi)$, and the elements of C are mutually incompatible, we have $\omega \models \varphi \wedge \neg\psi$ or $\omega \models \varphi \wedge \psi$ or $\omega \models \neg\varphi \wedge \psi$. By Proposition 6, it follows that $\varphi \vee \psi \mathrel{|\!\sim}^* \neg\varphi \wedge \psi$ or $\varphi \vee \psi \mathrel{|\!\sim}^* \varphi \wedge \psi$ or $\varphi \vee \psi \mathrel{|\!\sim}^* \varphi \wedge \neg\psi$.

(OC): By $\varphi \vee \psi \mathrel{|\!\sim}^* \varphi$, there is an $\omega \models \varphi$ maximal in $C(\varphi \vee \psi)$. By $\psi \vee \chi \mathrel{|\!\sim}^* \psi$, there is an $\omega' \models \psi$ maximal in $C(\psi \vee \chi)$. Since $\omega \models \varphi$, $\omega \in C(\varphi \vee \chi)$. Since $\omega' \models \psi$, $\omega' \in C(\varphi \vee \psi)$, hence $\omega \geq \omega'$. Hence ω is maximal in $C(\varphi \vee \chi)$, hence $\varphi \vee \chi \mathrel{|\!\sim} \varphi$. Since $\chi \in L^*$, $\varphi \vee \chi \mathrel{|\!\sim}^* \varphi$. QED

3.2 Representation results (qualitative)

In this section, we show that an inference relation with the properties listed in the Propositions 5 and 8 can be represented by the presumptively valid arguments of a case model. The cases of the representing case model are the extensions of the inference relation, i.e., those valid consequences that are logically maximally specific:

Definition 9 *Let* $\mathrel{|\!\sim} \subseteq L \times L$*, and* φ *and* $\omega \in L$*. Then* ω *expresses an* extension *of* φ *when:*

1. $\varphi \mathrel{|\!\sim} \omega$*; and*
2. $\omega \models \varphi$*; and*
3. *For all* $\psi \in L$*, if* $\varphi \mathrel{|\!\sim} \psi$ *and* $\psi \models \omega$*, then* $\omega \models \psi$*.*

Proposition 10 *Let* $\mathrel{|\!\sim} \subseteq L \times L$ *have the property (Ant) (as in Proposition 5). Then, for all* $\varphi \in L$*, if* $\varphi \mathrel{|\!\sim} \varphi$*, there is an* $\omega \in L$ *that expresses an extension of* φ*.*

Proof. Consider the set of sentences $S = \{\psi \mid \varphi \mathrel{|\!\sim} \psi\}$ and pick a sentence $\omega \in S$ that is logically maximally specific. Such a sentence exists since S is not empty (as $\varphi \in S$) and L is assumed to be generated by finitely many propositional constants, hence has a finite number of logical equivalence classes. We show that ω is an extension of φ (Definition 9): (i) $\varphi \mathrel{|\!\sim} \omega$ since $\omega \in S$. (ii) By (Ant), $\varphi \mathrel{|\!\sim} \varphi \wedge \omega$; hence $\varphi \wedge \omega \in S$. Since ω is maximally specific in S, it follows that $\omega \models \varphi$. (iii) Consider $\psi \in L$ for which $\varphi \mathrel{|\!\sim} \psi$ and $\psi \models \omega$. Then $\psi \in S$. Since ω is maximally specific in S, $\omega \models \psi$. QED

We define the counterpart of the logical algebra L^* used in Proposition 8.

Definition 11 *Let $\mid\!\sim \subseteq L \times L$ with the property (Ant) (as in Proposition 5), and $C \subseteq L$ the set of sentences expressing extensions. Then L^* denotes the closure of C under negation, conjunction and logical equivalence, and $\mid\!\sim^*$ the restriction of $\mid\!\sim$ to L^*.*

We can now formulate the representation theorem.

Theorem 12 *Let $\mid\!\sim \subseteq L \times L$ have the following properties:*

(LE) *If $\varphi \mid\!\sim \psi$, $\models \varphi \leftrightarrow \varphi'$ and $\models \psi \leftrightarrow \psi'$, then $\varphi' \mid\!\sim \psi'$;*
(Cons) *$\varphi \not\mid\!\sim \bot$;*
(Ant) *If $\varphi \mid\!\sim \psi$, then $\varphi \mid\!\sim \varphi \wedge \psi$;*
(RW) *If $\varphi \mid\!\sim \psi \wedge \chi$, then $\varphi \mid\!\sim \psi$;*
(CCM) *If $\varphi \mid\!\sim \psi \wedge \chi$, then $\varphi \wedge \psi \mid\!\sim \chi$;*
(CCT) *If $\varphi \mid\!\sim \psi$ and $\varphi \wedge \psi \mid\!\sim \chi$, then $\varphi \mid\!\sim \psi \wedge \chi$;*
(Coh) *$\varphi \mid\!\sim \varphi$ if and only if $\exists \varphi^* \in L^*$ with $\varphi^* \not\models \bot$ and $\varphi^* \models \varphi$;*
(Ch) *If $\varphi \mid\!\sim^* \varphi$ and $\psi \mid\!\sim^* \psi$, then $\varphi \vee \psi \mid\!\sim^* \neg\varphi \wedge \psi$ or $\varphi \vee \psi \mid\!\sim^* \varphi \wedge \psi$ or $\varphi \vee \psi \mid\!\sim^* \varphi \wedge \neg\psi$;*
(OC) *If $\varphi \vee \psi \mid\!\sim^* \varphi$ and $\psi \vee \chi \mid\!\sim^* \psi$, then $\varphi \vee \chi \mid\!\sim^* \varphi$.*

Then there is a case model $(C, \geq)$ with the property:

$\varphi \mid\!\sim \psi$ if and only if $(C, \geq) \models \varphi \leadsto \psi$.

Proof. Given $\mid\!\sim$ with the properties mentioned, we consider the set of all extension expressions $E := \{\omega \in L \mid \exists \varphi \in L : \omega \text{ expresses an extension of } \varphi\}$. Let C be a set containing one element for each logical equivalence class in E. For ω and $\omega' \in C$, define $\omega \geq \omega' := \omega \vee \omega' \mid\!\sim \omega$. We show that the pair $(C, \geq)$ is a case model with the property of the theorem.

(i) *$(C, \geq)$ is a case model.*

We check the properties of the definition of a case model. 1. Let $\varphi \in C$. Then $\varphi \mid\!\sim \varphi$. If $\models \neg\varphi$, by (LE) and (RW), $\varphi \mid\!\sim \bot$, contradicting (Cons).

2 & 3. Consider φ and $\psi \in C$. Then $\varphi \mid\!\sim^* \varphi$ and $\psi \mid\!\sim^* \psi$. By (Ch), $\varphi \vee \psi \mid\!\sim^* \neg\varphi \wedge \psi$ or $\varphi \vee \psi \mid\!\sim^* \varphi \wedge \psi$ or $\varphi \vee \psi \mid\!\sim^* \varphi \wedge \neg\psi$. Since φ and ψ are extensions, when $\varphi \vee \psi \mid\!\sim^* \neg\varphi \wedge \psi$, $\varphi \models \neg\psi$. When $\varphi \vee \psi \mid\!\sim^* \varphi \wedge \psi$, $\varphi \models \psi$ and $\psi \models \varphi$, so $\varphi = \psi$. When $\varphi \vee \psi \mid\!\sim^* \varphi \wedge \neg\psi$, $\psi \models \neg\varphi$.

4. Consider φ and $\psi \in C$. By (Ch), we have three cases: When $\varphi \vee \psi \mid\!\sim^* \neg\varphi \wedge \psi$, by (RW): $\varphi \vee \psi \mid\!\sim^* \psi$, i.e., $\psi \geq \varphi$. When $\varphi \vee \psi \mid\!\sim^* \varphi \wedge \psi$, by (RW): $\varphi \vee \psi \mid\!\sim^* \varphi$, i.e., $\varphi \geq \psi$ (and in this case also $\psi \geq \varphi$). When $\varphi \vee \psi \mid\!\sim^* \varphi \wedge \neg\psi$, by (RW): $\varphi \vee \psi \mid\!\sim^* \psi$, i.e., $\varphi \geq \psi$.

5. Consider φ, ψ and $\chi \in C$. Assume $\varphi \geq \psi$ and $\psi \geq \chi$. In other words, by the definitions, $\varphi \vee \psi \mid\!\sim^* \varphi$ and $\psi \vee \chi \mid\!\sim^* \psi$. Then (OC) gives $\varphi \vee \chi \mid\!\sim^* \varphi$, i.e., $\varphi \geq \chi$.

(ii) *If $\varphi \mid\!\sim \psi$, then $(C, \geq) \models \varphi \leadsto \psi$.*

Assume $\varphi \mid\!\sim \psi$. Then, by (Ant), $\varphi \mid\!\sim \varphi \wedge \psi$. By (CCM), $\varphi \wedge \psi \mid\!\sim \varphi \wedge \psi$. By Proposition 10, there is an $\omega \in C$ that is an extension of $\varphi \wedge \psi$. In particular, $\omega \models \varphi \wedge \psi$ and $\omega \in C(\varphi)$. Let $\omega' \in C(\varphi)$. Then $\omega \models \omega \vee \omega'$ and $\omega \vee \omega' \models \varphi$. Hence, by $\varphi \mid\!\sim \omega$ and (CCM), $\omega \vee \omega' \mid\!\sim \omega$, i.e., $\omega \geq \omega'$. In other words, ω is maximal in $C(\varphi)$.

(iii) *If $(C, \geq) \models \varphi \leadsto \psi$, then $\varphi \mid\!\sim \psi$.*

By definition, if $(C, \geq) \models \varphi \leadsto \psi$, there is an $\omega \in C$ with $\omega \models \varphi \wedge \psi$, maximal in

$C(\varphi)$. Hence $\omega \models \varphi$ and, by (Coh), $\varphi \mid\!\sim \varphi$. By Proposition 10, there is an $\omega' \in C(\varphi)$ that expresses an extension of φ. So $\varphi \mid\!\sim \omega'$. By (RW) (and (LE)), $\varphi \mid\!\sim \omega \vee \omega'$. Since ω is maximal in $C(\varphi)$, $\omega \geq \omega'$, i.e., $\omega \vee \omega' \mid\!\sim \omega$. By (CCT), it follows that $\varphi \mid\!\sim \omega$. So ω also expresses an extension of φ. Since $\omega \models \psi$, (RW) gives $\varphi \mid\!\sim \psi$. QED

3.3 Representation results (quantitative)

In this section, we show that our notion of presumptively valid inference can also be quantitatively represented. We use the following lemma that the preference relations of our case models are exactly those that can be numerically represented.

Lemma 13 *Let $C \subseteq L$ be finite with elements that are logically consistent, mutually incompatible and different (properties 1, 2 and 3 in the definition of case models). Then the following are equivalent:*

1. *$(C, \geq)$ is a case model;*
2. *$\geq$ is numerically representable, i.e., there is a real valued function v on C such that for all φ and $\psi \in C$, $\varphi \geq \psi$ if and only if $v(\varphi) \geq v(\psi)$.*

The function v can be chosen with only positive integer values.

Proof. It is a standard result in order theory that total preorders on countable sets are the ones that are representable by a real-valued function [23]. In our finite setting, the numbers can be chosen as positive integer values. QED

Definition 14 *Let $(C, \geq)$ be a non-empty case model and v a positive numeric function that represents $\geq$. Then we define, for all φ and $\psi \in L$:*

1. $v(\varphi) := \max\{v(\omega) \mid \omega \in C, \omega \models \varphi\}$*;*
2. $w(\varphi) := \sum\{v(\omega) \mid \omega \in C, \omega \models \varphi\}$*;*
3. $s(\varphi) := w(\varphi)/w(\top)$*;*
4. $s(\varphi, \psi) := w(\varphi \wedge \psi)/w(\varphi) = s(\varphi \wedge \psi)/s(\varphi)$ *(with $s(\varphi) > 0$);*
5. $v(\varphi, \psi) := v(\varphi \wedge \psi)$.

We say that $v(\varphi)$ is the value *of φ and $w(\varphi)$ its* weight. *We say that $s(\varphi, \psi)$ is the* strength *of the argument from φ to ψ, and $v(\varphi, \psi)$ its* value.

Corollary 15 *Let L^* denote the closure of C under negation, conjunction and logical equivalence. Then the function s restricted to L^* obeys the axioms of probability functions, i.e., for all φ and $\psi \in L^*$:*

1. $s(\varphi) \geq 0$*;*
2. $s(\top) = 1$*;*
3. *If $\varphi \wedge \psi \models \bot$, then $s(\varphi \vee \psi) = s(\varphi) + s(\psi)$.*

The coherence and conclusiveness of arguments can be represented in terms of these numeric functions, as in the following theorems.

Theorem 16 *(Coherence) Let $(C, \geq)$ be a non-empty case model and v and s as above. Then the following are equivalent, for all φ and $\psi \in L$:*

1. *The argument from φ to ψ is coherent;*
2. $v(\varphi \wedge \psi) > 0$*;*
3. $s(\varphi \wedge \psi) > 0$*.*

Proof. An argument is coherent if and only there is a case implying the case made by the argument. This is exactly so when the case made has positive value. This is equivalent to the strength of the argument having positive value. QED

Theorem 17 *(Conclusiveness) Let $(C, \geq)$ be a non-empty case model and w and s as above. Then the following are equivalent, for all φ and $\psi \in L$:*

1. *The argument from φ to ψ is conclusive;*
2. $w(\varphi \wedge \psi) = w(\varphi) > 0$*;*
3. $s(\varphi, \psi) = 1$*.*

Proof. An argument is conclusive if and only if it is coherent and all cases implying the premises also imply the conclusions. This is exactly so when the cases implying the premises coincide with the cases implying the case made by the argument, i.e., when the weights of premises and case made are equal. That is exactly the case when the argument's strength is equal to 1. QED

The next theorem characterizes presumptive validity using a value function v. We restrict to L^*.

Theorem 18 *(Presumptive validity; in terms of value) Let $(C, \geq)$ be a non-empty case model and L^*, v as above. Then the following are equivalent, for all φ and $\psi \in L^*$:*

1. *The argument from φ to ψ is presumptively valid;*
2. $v(\varphi \wedge \psi) = v(\varphi)$*.*

Proof. An argument is presumptively valid if and only if there is a case implying the case made by the argument that is at least as preferred as all cases implying the premises. This is exactly so when the value of $\varphi \wedge \psi$ is equal to that of φ. QED

In order to characterize presumptive validity in terms of a strength function s, we choose the value function from which it is derived with special care, as in this lemma:

Lemma 19 *Let α be a positive number. Then the function v in Definition 14 can be chosen such that, for all $\omega \in C$:*

$v(\omega) > (\alpha + 1)w(\kappa)$*, where* $\kappa := \vee\{\omega^* \in C \mid \omega > \omega^*\}$*.*

We say that v is α-separating.

Theorem 20 *(Presumptive validity; in terms of strength) Let $(C, \geq)$ be a non-empty case model and α the maximal number of elements in an equivalence class of the preference relation. Let L^*, v, w and s be as above, with v α-separating (as in the lemma). Then the following are equivalent, for all φ and $\psi \in L^*$:*

1. *The argument from φ to ψ is presumptively valid;*
2. $s(\varphi, \psi) > 1/(\alpha + 1)$*.*

Proof. From 1 to 2: Let the argument from φ to ψ be presumptively valid. Then (and only then) the values of φ and $\varphi \wedge \psi$ are equal to the value $v(\omega)$ of a case $\omega \in C$ that is an extension of φ. The case ω can be chosen such that it implies the case made by the argument, but that makes no difference for the value $v(\omega)$. Let κ denote the disjunction of all cases of value smaller than $v(\omega)$ (cf. the use of κ in the lemma). We have the following inequalities:

$$\begin{aligned}\frac{w(\varphi \wedge \psi)}{w(\varphi)} &\geq \frac{v(\omega)}{\alpha v(\omega) + w(\kappa)} \\ &> \frac{v(\omega)}{\alpha v(\omega) + v(\omega)/(\alpha+1)} = \frac{(\alpha+1)v(\omega)}{\alpha(\alpha+1)v(\omega) + v(\omega)} \\ &= \frac{(\alpha+1)}{\alpha(\alpha+1)+1} > \frac{(\alpha+1)}{\alpha(\alpha+1)+(\alpha+1)} = \frac{1}{\alpha+1}.\end{aligned}$$

From 2 to 1: Let the argument from φ to ψ be presumptively invalid. Then (and only then) the value of φ, say $v(\omega)$ for a case $\omega \in C$, is higher than the value of $\varphi \wedge \psi$. Let κ be as before. Then we have these inequalities, completing the proof:

$$\frac{w(\varphi \wedge \psi)}{w(\varphi)} \leq \frac{w(\kappa)}{v(\omega)} < \frac{v(\omega)/(\alpha+1)}{v(\omega)} = \frac{1}{\alpha+1}. \quad \text{QED}$$

4 Discussion and conclusion

We set out to answer the semantics and normative questions for reasoning with presumptive arguments: How are presumptive arguments grounded in interpretations; and when are they evaluated as correct? Our formalism answers these questions, as follows.

As to the semantics question, we have proposed to interpret arguments in models consisting of cases and their preferences. Cases are structured expressions of what can be the case, formalized as consistent sentences in a classical logical language. Our cases are akin to exemplars, observations, precedents, situations, prototypes, schemes, scenarios, scripts, and other structured representations of parts of the world we live in. Our cases are to be contrasted with formal models or worlds that represent completely specified representations, in the sense that all properties are evaluated (as used for instance in preferential model semantics [16]). Key properties of our definition of cases are their logical consistency and mutual incompatibility. Hence, our cases can be thought of as distinguishable coherent combinations of properties. In a case model, the cases come with a preference relation expressing their relative value. Such values can be interpreted objectively, for instance as derived from frequencies or probabilities. However there is no reason to restrict to objective interpretations, and one can also think of subjective values, e.g., derived from utilities, as used in theories of decision making and preference-based choice.

The normative question is answered in terms of this case semantics. We have distinguished coherent, presumptively valid and conclusive arguments. For coherent arguments, there must be a case that implies the case made by the argument. For presumptively valid arguments, there must be a case implying the case made by the argument that is of at least as high value as the other cases implying the argument's premises. For conclusive arguments, there must be a case implying the case made by the argument, and all cases implying the premises should imply the argument's conclusions. When a presumptively valid argument is not conclusive, the argument is defeasible.

We have shown qualitative properties that characterize presumptively valid arguments. We have proven that coherence, presumptive validity and conclusiveness also can be defined in terms of quantitative interpretations. In particular, we have shown a characterization of presumptively valid arguments in terms of two kinds of numeric functions. The first (used in Theorem 18) is a value function v that maximizes (instead of sums) the values of cases. The second (used in Theorem 20) is an argument strength function s that obeys the probability axioms. This provides a probabilistic representation of presumptive validity, which is interesting now that our proposal does not exclude an interpretation of arguments for subjective, preference-based choices, more typically associated with utility functions. The value function v maximizes instead of sums the values of cases, hence reminds of how possibilistic logic [10] contrasts with probabilistic approaches (see also [29]).

By these answers to the semantics and normative questions, we have provided a theory of presumptive arguments with close ties to classical logic and standard probability theory. In contrast, in his influential work on argumentation in artificial intelligence [20, 21], Pollock argued against approaches based on classical logic and standard probability theory. Before him, the philosopher and argumentation theorist Toulmin argued similarly [27], without developing an alternative formal and computational perspective, as did Pollock.

[6] focus on the development of the theory of deductive arguments, while our proposal emphasises ampliative argument in their relation to deductive (here: conclusive) and defeasible (here: presumptively valid) arguments. Compared to [19], who provide a framework of argumentation with preferences that can be instantiated with different abstract argumentation semantics, the present proposal gives a model-based formal semantics leading to a formal definition of presumptive validity. Connections between argumentation and uncertain reasoning have been investigated [14, 15, 17, 26], several focusing on abstract argumentation. Our proposal has a model-based definition of presumptive validity that comes with both a qualitative and a quantitative interpretation.

This work connects semantics and properties of inference relations (cf. the research program proposed by [25]; see also [1]). [2] discuss the postulates closure, direct consistency and indirect consistency for the evaluation of argumentation formalisms. Analogs of these properties obtain for presumptively valid arguments. A property akin to closure under strict rules is that a presumptively valid argument remains valid when conclusive consequences of the case made are added. A property related to direct and indirect consistency is that in the present proposal extensions are consistent. A question that arises is how the present proposal is formally connected to related formalisms. In particular, it is natural to study connections with preferential modal logics. Also the place of this work among other studies of nonmonotonic inference relations [18] is a relevant topic of further study, in particular also [7].

The case model semantics presented here formalizes ideas semi-formally presented in [31]. That work was inspired by research using Bayesian networks for modeling argument-based and scenario-based reasoning with evidence [32], where well-known issues with Bayesian network modeling were encountered, namely first that such modeling typically requires many more numbers than are reasonably available, and second that—notwithstanding their transparent formal definition—Bayesian networks are easily misinterpreted, e.g., in causal terms (cf. [9]). In contrast with Bayesian networks, the present formalism is probabilistic, but does not require many numbers, and provides a formal interpretation of arguments in case models.

In conclusion, the present paper has contributed a logic of presumptively valid arguments using case models as semantics. The resulting formalism models correct grounded reasoning with presumptive arguments. As such, we have provided a perspective on how to formally combine logic, probability theory and argumentation, suggesting applications that require the representational power of logic, the data-analytic strength of probability theory, and the interactive social construction of argumentation.

By the combination of logical and probabilistic modeling primitives using an argumentation perspective, the present proposal is a step in the much-needed unification of logic and probability in AI [24].

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