Neighbourhood Structure in Large Games

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ABSTRACT

We study repeated normal form games where the number of players is large and suggest that it is useful to consider a neighbourhood structure on the players. The structure is given by a graph G whose nodes are players and edges denote visibility. The neighbourhoods are maximal cliques in G . The game proceeds in rounds where in each round the players of every clique X of G play a strategic form game among each other. A player at a node v strategises based on what she can observe, i.e., the strategies and the outcomes in the previous round of the players at vertices adjacent to v. Based on this, the player may switch strategies in the same neighbourhood, or migrate to another neighbourhood. Player types, giving the rationale for such switching, are specified in a simple modal logic.

We show that given the initial neighbourhood graph and the types of the players in the logic, we can effectively decide if the game eventually stabilises. We then prove a characterisation result for these games for arbitrary types using potentials. We then offer some applications to the special case of weighted co-ordination games where we can compute bounds on how long it takes to stabilise.

Categories and Subject Descriptors

F.4 [**Mathematical logic and formal languages**]: Modal logic, temporal logic

General Terms

Theory

Keywords

Game dynamics, neighbourhood structure, type specification, potential.

1. SUMMARY

In Indian towns, it is still possible to see vegetable sellers who carry vegetables in baskets or pushcarts and set up shop in some neighbourhood. The location of their 'shop' changes dynamically, based on the seller's perception of demand for vegetables in different neighbourhoods in the town, but also on who else is setting up shop near her, and on her perception of how well these (or other) sellers are doing. Indeed, when she buys a lot of vegetables in the wholesale market, the choice of her 'product mix' as well as her choice of location are determined by a complex rationale. While the prices she quotes do vary depending on general market situation, the neighbourhoods where she sells also influence the prices significantly: she knows that in the poorer neighbourhoods, her buyers cannot afford to pay much. She can be thought of as a small player in a large game, one who is affected to some extent by play in the entire game, but whose strategising is local where such locality is itself dynamic.

In the same town, there are other, relatively better off vegetable sellers who have fixed shops. Their prices and product range are determined largely by wholesale market situation, and relatively unaffected by the presence of the itinerant vegetable sellers. If at all, they see themselves in competition only against other fixed-shop sellers. They can be seen as big players in a large game.

What is interesting in this scenario is the movement of a large number of itinerant vegetable sellers across the town, and the resultant increase and decrease in availability of specific vegetables as well as their prices. We can see the vegetable market as composed of dynamic neighbourhoods that expand and contract, and the dynamics of such a structure dictates, and is in turn dictated by the strategies of itinerant players.[1]

Such division into neighbourhoods need not be spatial or physical, but can also be logical. Consider, for instance, online stores such as Amazon, eBay, Yahoo Shopping, Rediff Shopping etc. Sellers put their items up for sale on one or more of these stores based on the demand of these items and the outcomes so far. A seller who puts her item up on eBay today may very well switch to Amazon tomorrow if the demand there is higher. The buyers, on their part, would generally want the best price on offer. Hence a buyer who bought an item from Amazon today might buy another of the same kind tomorrow from eBay.

We call games **large** when the number of players in them is so large that it is hard to consider any player to be playing against everyone else, and where a player may not even know the number of players in the game, let alone how each

[1] In fact, the movement of these sellers may further depend on the cost of transport between these places because of small profit margins.

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of them play. The market and the Internet are often cited as examples of large games. In such games, a flat structure of all players as "equals" hides important detail: neither does a player consider the detailed play of every other, nor does a player consider all other players to be of one 'average' type. We suggest that it is useful to group players into logical neighourhoods in such games: within a neighbourhood players strategise interpersonally; across neighbourhoods their visibility, and hence strategising, is limited. In the latter situation, heuristic play becomes significant. Moreover, game dynamics alters neighbourhood structure, and conversely.

Though we speak of dynamic neighbourhood structure, we note that static neighbourhood structure makes sense as well. For instance, consider the game of chess. A player can be a grandmaster, a national master, a professional or an amateur. It is generally the case that the grandmasters play among themselves, the national masters play each other and so on. Moreover the lesser non-professional players are also constrained by time, location, resources etc. Thus, for instance, a medium rated player in New Delhi would usually take part in tournaments in and around New Delhi. But how do these players strategise? The same medium rated player may not be able to take part in a tournament in Moscow (say), but that doesn't prevent her from following what is going on in that particular tournament. If a player in the tournament in Moscow is faring well by playing the Hungarian defence, our player in New Delhi may well employ the same strategy in her tournament in the hope of doing better.

It can be meaningfully argued that the games in New Delhi, Dortmund and Wijk an Zee are all subgames in one large game, in the sense that strategising and play in one is influenced by play in the other and become part of communal memory. Once again, a neighbourhood substructure abstracts such influence in the large game.

Similar structuring is seen in many other games. In football, for instance, every team all over the world participates only in three or four different leagues each year: the English Premier League, La Liga, Serie A, Bundesliga etc. But every team closely follows the unfolding of play in the other leagues and strategises based not only on the outcomes of its own league but also on those of the others. Here again, the neighbourhoods may change dynamically as the game progresses. These changes are brought about by teams/players switching allegiances. A player playing in league 1 today may think that his strategy and style of play is more suited for a different league and that he can do much better there. Hence he might join the latter league tomorrow.

In this paper, we study large games in which players are arranged in certain neighbourhoods. The neighbourhood structure is given by a finite undirected graph where the vertices of the graph represent players and edges represent their visibility. The cliques in the graph represent the different neighbourhoods of players. We prove a characterization theorem on such games. Then we study weighted coordination anonymous games. We consider both the variations: neighbourhood structures that are static as well as the dynamic ones (where the neighbourhood structure changes after every round).

Our model is that of an infinite repeated game. In every round every player plays a strategic form game with the players of her clique. The players are among a fixed set of types, which determine their strategies. In every strategic form game in a neighbourhood, the payoffs of the players are determined by the action profile of the players of that neighbourhood in that particular round.

We are interested in the dynamics of such games and their eventual stability. What action profiles, strategies, configurations etc. eventually arise? We call a game eventually stable if eventually a set of configurations repeat cyclically forever (e.g. a set of localities in the town for the vegetable seller in an Indian town). This set might be a singleton in which case the actions of the players don't change anymore; the configuration is static. We are also interested in how long it takes for such a game to eventually stabilise. We show the following:

- We define a simple modal logic in which a player's rationale for type switching can be specified. When the types of the players are specified in this logic, we show that it can be effectively decided whether the game eventually stabilises.
- When the types of the players are unknown, we show that one can associate a potential with every configuration such that the potential becomes constant if and only if the game eventually stabilises.
- We study an application to weighted co-ordination games and explore the consequences when the players play simple imitative strategies. We show that in such cases the game always stabilises and one can compute an upper bound on the number of rounds needed to attain stability.

A valid objection at this point, at least in the case of static neighbourhoods, is the following. If the players of every neighbourhood play normal form games in every round among themselves why is it not the case that the expected outcome is a Nash equilibrium of every normal form game in every neighbourhood? There are two explanations for this. First, since the model is that of repeated normal form, there might be action tuples other than the Nash equilibrium tuples that are in equilibrium (as for example in tit-for-tat in repeated prisoners' dilemma). But the more potent argument is that when the game is large, players hardly have the expertise, knowhow or even resources to compute and play the Nash equilibrium tuple. They play based on heuristics and employ simple strategies such as imitation, tit-for-tat, follow-the-leader etc. Hence the outcome may be much more varied than the Nash equilibrium tuples. Thus, we feel that a more natural question to ask in the setting of large games is on the dynamics of the game given the types of the players and their eventual stability and also on what configurations eventually arise. See [14, 13, 12] for more on this.

Related Work

Analysis of eventual dynamics in games is of course not new in game theory. In the study of dynamical systems, researchers analyse games where the dynamics is given by certain differential equations and the various parameters of the equation control the dynamics. The solution to the equation gives the stable behaviour. Evolutionary game theory [18, 17] studies how games evolve by the successive elimination of dominated strategies and the emergence of evolutionary stable strategies.

We study games where the players are represented by the vertices of a graph and the edges of the graph give the other players they can interact with. Such games have been studied, for instance, in [10, 9, 7, 8]. They analyse games where the payoff of players depend only on her own action and the actions of her adjacent players as given by the graph structure. They study the existence and computation of Nash equilibria in such games. Young ([15, 16]) studies how innovations spread through society by observation and interaction. He too models the interaction structure of the players by a finite undirected graph. Imitation dynamics in congestion games have been studied, for instance, in [2, 1], where they study asymptotic time complexities of the convergence or non-convergence to Nash equilibrium in congestion games when the players play imitative strategies.

In the weighted co-ordination games we study in this paper, in every round and in every neighbourhood, the payoffs of the strategic form games do not depend on the actual action profile of the players in the neighbourhood but only on the distribution of the actions. As the actions come from a common set, such a distribution, in every round, is well-defined and non-trivial. Games where the payoffs depend on the action distributions of the players are called anonymous games and have been extensively studied in the literature. See, for instance, [3, 4, 5, 6] and the references therein.

2. PRELIMINARIES

Let $N = \{1, 2, \ldots, n\}$ be the set of players. The players are arranged in a neighbourhood structure given by a simple undirected graph $G = (V[G], E[G])$ without self loops called the neighbourhood graph. Every vertex of G stands for a player and we use the letters i, j, k etc. to denote both the vertices of the graph and players from N. The neighbourhood graph G is topologically described as follows.

Let $clq[G]$ be the set of maximal cliques of G. For simplicity, we assume that the maximal cliques are non-overlapping, that is, every vertex $i \in V[G]$ is part of exactly one maximal clique. The results of this paper go through even we drop this assumption. These maximal cliques are the neighbourhoods of the players. Moreover, a vertex i in any clique X may have edges to vertices in some other clique X'. For a player, i these edges give the visibility structure of i. Thus the player i can view the moves and outcomes of all the players that are in her clique and also that of some players from other neighbourhoods.

Let $A = \{a_1, a_2, \ldots, a_{|A|}\}$ denote the common set of actions of all the players. Given a neighbourhood graph G, we denote by $X[G](i)$ the maximal clique (neighbourhood) that player i belongs to. As usual, we let $iE[G] = \{j \mid (i, j) \in E[G]\}$ be the set of vertices adjacent to i, that is $iE[G]$ is the set of players visible to player i. Note that $iE[G] \cap X[G](i) = X[G](i) \setminus \{i\}$. Let $nbd[G](i)$ be the set of neighbourhoods visible to player i. These are the neighbourhoods, at least one player of which i can view. Thus $nbd[G](i) = \{X[G](j) \mid j \in iE[G]\}$. See Figure 1 for an example.

The type of a player specifies how she strategises. These are functions that will be defined below, but we assume a set Γ of player types, and a type map $typ : N \to \Gamma$. As a rule, $|\Gamma| << |N|$, reflecting the intuition that in a large game, the number of players may be large but there are only a few player types. We use γ, γ' etc. to range over Γ, and specify typ by an n-tuple $\langle \gamma_1, \ldots, \gamma_n \rangle$.

Next, we need to talk about the outcomes of the games. For that, rather than work with payoffs, we use a propositional language. Fix $\mathcal{P}$, a countable set of atomic propositions. $\mathcal{P}$ consists of propositions which stand for statements of the form:

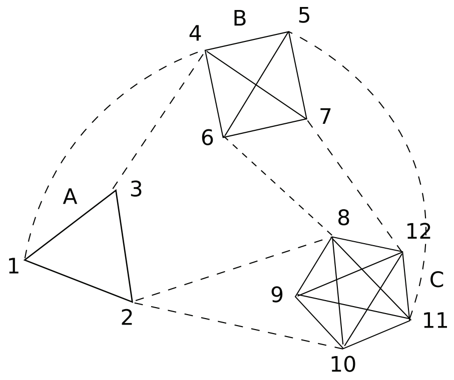

Figure 1: A neighbourhood graph. A, B, C **are the neighbourhoods (maximal cliques in the graph) and** $1, 2, \ldots, 12$ **are the players (vertices)**

- action a is played,
- payoff is greater than a threshold c,
- payoff is greater than all neighbours,

and so on. Every game will involve only a finite set $P \subseteq \mathcal{P}$ of these propositions. The game proceeds in rounds. In every round k, the players of every neighbourhood play a normal form game among themselves. The outcome of the entire game in that round k, is thus the outcomes of these normal form games.

Since the games are large (both in terms of the number of players and their logical structure), it is natural to assume that the outcome in any round does not depend on the identity of the players or the profile of actions played by them. Rather in any round k, given the neighbourhood graph G_k for that round, the payoffs of the players depend only on the distribution of the actions in the various neighbourhoods given by G_k. Such games are called anonymous games [3, 4, 5, 6].

An action distribution for a neighbourhood X of size k is an $|A|$ tuple of integers $\mathbf{y} = (y_1, \ldots, y_{|A|})$ such that $y_j \geq 0$ and $\Sigma_{j=1}^{k} y_j = k,\ 1 \leq j \leq |A|$. That is, the jth component of $\mathbf{y}$ gives the number of players in the neighbourhood X who play action a_j. Let $\mathbf{Y}[k]$ denote the set of all action distributions of a neighbourhood of size k and let $\mathbf{Y} = \bigcup_{k=1}^{n} \mathbf{Y}[k]$.

We have an outcome function $out : \mathbf{Y} \to 2^P$ which gives the truth of the outcome propositions P at any neighbourhood X of size k according to the action distribution of the players of that neighbourhood.

Now given a neighbourhood graph G, we can lift out to a valuation function at the vertices of G: $val_{out}[G] : N \to 2^P$ $val_{out}[G](i)$ gives the truth of the propositions which talk about the outcomes of $\{i\} \cup nbd[G](i)$.

Thus formally, a game $\mathcal{G}$ is a tuple $\mathcal{G} = (typ, P, out)$, where typ is a type map, P a subset of $\mathcal{P}$ and out an outcome function. A configuration of the game is a pair $c = (G, \mathbf{a})$ where G is a neighbourhood graph and $\mathbf{a} \in A^n$ is an action profile. Let C be the set of all configurations. Note that the size of C, that is, the total number of configurations, is $\binom{n}{2} \times |A|^n$.

When an initial configuration c_0 is specified, we call the pair $(\mathcal{G}, c_0)$ an initialised game. A play (history) in an initialised game $(\mathcal{G}, c_0)$, where $c_0 = (G_0, \mathbf{a}_0)$, is a sequence $\rho = (G_0, \mathbf{a}_0), \ldots, (G_k, \mathbf{a}_k),\ k > 0$, of configurations, where for all $i \geq 0$, $V[G_i] = V[G_0]$. Let H denote the set of all histories. We call a game **static neighbourhood** if, in every history in H, for all $i \geq 0$, $G_i = G_0$; otherwise it is a **dynamic neighbourhood** game.

Given a neighbourhood graph G, and a player $i \in N$, a choice for player i is a pair (X, a) where $X \in nbd[G](i)$ and $a \in A$. Let $\chi[G](i)$ denote the set of choices of i in G. A type γ is then a map $\gamma : H \to 2^{(2^N \times A)}$ such that for all $\rho \in H$, $\gamma(\rho) \subseteq \chi[G](i)$ where $G = G_{|\rho|}$.

Remark Note that the notion of types in our setting is a bit different from the notion of strategies in the classical sense. The type of a player prescribes her a set of choices after every history whereas, a strategy usually prescribes a unique choice. In the case of types, the action that the player plays is a non-deterministic choice from this set.

In a static neighbourhood game, players cannot switch neighbourhoods between rounds (but can switch strategies). Thus in a static neighbourhood game, given a neighbourhood graph G, if $(X, a) \in \chi[G](i)$, then $X = X[G](i)$. However, in a dynamic neighbourhood game, a player i can decide to move to a different neighbourhood from round k to round $k+1$ provided the new neighbourhood is in $nbd[G](i)$. Thus in the dynamic neighbourhood game, the underlying neighbourhood graph keeps changing.

We say that a history $\rho = (G_0, \mathbf{a}_0), \ldots, (G_m, \mathbf{a}_m)$ is coherent with respect to the player types $\langle \gamma_1, \ldots, \gamma_n \rangle$ if the following conditions hold. Let ρ_k be the length k prefix of ρ. Then for every $k :\ 0 \leq k < m$:

- For every $i \in N$, if $\mathbf{a}_{k+1}(i) = a$ then $(X, a) \in \gamma_i(\rho_k)$.
- Given that $G_k = (V, E_1)$ and $G_{k+1} = (V, E_2)$ there exists a choice tuple $\langle (X_1, a_1), \ldots, (X_n, a_n) \rangle$ such that for all i, $(X_i, a_i) \in \gamma_i(\rho_k)$, and for all $l, m \in N$:
  - $(l, m) \in E_2 \setminus E_1$ implies $m \in X_l$, and
  - $(l, m) \in E_1 \setminus E_2$ implies $m \notin X_l$.

In other words, every player i that joins a neighbourhood X in round $k + 1$, has an edge in the neighbourhood graph G_{k+1} to all the other players who also decide to join (or stay put) in the same neighbourhood X in round $k + 1$. In addition, her visibility structure changes, in that, she may be able to view the outcomes and actions of new players in some neighbourhood other than X after joining X whereas, some of the players that she could view in round k, may not be visible to her anymore. Note that the process is non-deterministic. See figure 2 for an example.

Some typical types of players are

- Play the action played by the maximum number of visible players in the previous round.
- Play the action played by the player who, among the visible ones, received the maximum payoff in the last round.
- Play the action played by the player who, among the visible ones in the last round, received the maximum average payoff in the previous k rounds.

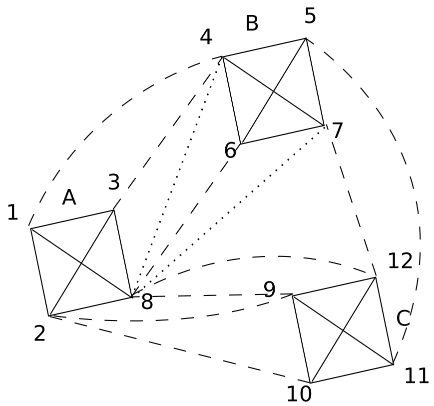

Figure 2: The neighbourhood graph of figure 1 after player 8 has joined the neighbourhood of players 1,2 and 3. The dashed edges are the visibility of player 8 retained from her old neighbourhood and the dotted ones are the players newly visible to her.

- Switch from the current neighbourhood to a neighbourhood where a player of the same type received a higher payoff in the previous round.

and so on.

Let $c, c' \in C$ be configurations. c' is said to follow c, denoted $c \to c'$ if c' is derived from c as above (that is, when all players play according to their types specified by the game). Let $\to^*$ be the transitive closure of the follows relation. The graph $\mathcal{C} = (C, \to)$ will be referred to as the configuration graph of the game. Let $c \in C$; we then speak of $\mathcal{T}_{\mathcal{C}}(c)$, the tree unfolding of the configuration graph from c defined in the standard manner. $\mathcal{T}_{\mathcal{C}}(c) = (T, E)$ is an infinite tree where the nodes are labelled with configurations from C. For a node $t \in T$, we let $c(t)$ denote this configuration.

3. TYPES

We have spoken of players switching strategies or migrating to other neighbourhoods. What is the rationale for such switching? In general, this is to improve payoffs over the course of play. In large games, since players have only a partial knowledge of the game, the rationale is likely to be based on a host of heuristic strategies, like "follow the leader", "imitate the player with the maximum payoff" and so on. These strategies, though suboptimal, are easy to employ and switch between, and have the property that in most cases, they do not perform much worse than the optimal strategies. See [14, 13, 12] for more on this. Rather than consider a specific heuristic, we find it convenient to introduce a logical language to talk about the types of the players. The syntax should be able to specify the properties of the games that the players observe and the actions they play based on these observations.

Let X be a countable set of variables. Let the terms of the logic be defined as

$$\tau ::= i \mid x,\ i \in N,\ x \in X$$

That is, a term is either a player (vertex) or a variable (which takes players as its values). Then the types of the players are built using the following syntax:

$$\Phi ::= \tau_1 = \tau_2 \mid \tau_1 \leftrightarrow \tau_2 \mid [\tau_1, \tau_2] \mid p@\tau, p \in \mathcal{P} \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \ominus\varphi \mid \bigcirc\varphi \mid \diamondminus\varphi \mid \Diamond\varphi \mid \exists x \cdot \varphi(x)$$

where τ_1 and τ_2 are terms.

Let $(\mathcal{G}, c_0)$ be an initialised game. The formulas in Φ are evaluated at the nodes of $\mathcal{T}_{\mathcal{C}}(c_0)$. The truth of a formula $\varphi \in \Phi$ at a node $t \in \mathcal{T}_{\mathcal{C}}(c_0)$ is denoted by $t \models \varphi$ and is defined inductively as follows. Let $c(t) = (G, \mathbf{a})$ be the configuration associated with t. The truth of the atomic formulas $\tau_1 = \tau_2, \tau_1 \leftrightarrow \tau_2$ and $[\tau_1, \tau_2]$ are derived from $c(t)$:

- $t \models \tau_1 = \tau_2$ iff $\pi(\tau_1) = \pi(\tau_2)$.
- $t \models \tau_1 \leftrightarrow \tau_2$ iff $(\pi(\tau_1), \pi(\tau_2)) \in E[G]$.
- $t \models [\tau_1, \tau_2]$ iff $\exists X \in clq[G]$ such that $\pi(\tau_1) \in X$ and $\pi(\tau_2) \in X$.

For the rest of the formulas, we define truth by:

- $t \models p@\tau$ iff $p \in val_{out}[G](\pi(\tau))$.
- $t \models \neg\varphi$ iff $t \not\models \varphi$.
- $t \models \varphi_1 \vee \varphi_2$ iff $t \models \varphi_1$ or $t \models \varphi_2$.
- $t \models \ominus\varphi$ iff t is not the root of $\mathcal{T}_{\mathcal{C}}(c_0)$ and $t' \models \varphi$ where t' is the parent of t in $\mathcal{T}_{\mathcal{C}}(c_0)$.
- $t \models \bigcirc\varphi$ iff there exists a child t' of t in $\mathcal{T}_{\mathcal{C}}(c_0)$ such that $t' \models \varphi$.
- $t \models \diamondminus\varphi$ iff there exists an ancestor t' of t in $\mathcal{T}_{\mathcal{C}}(c_0)$ such that $t' \models \varphi$.
- $t \models \Diamond\varphi$ iff there exists a successor t' of t in $\mathcal{T}_{\mathcal{C}}(c_0)$ such that $t' \models \varphi$.
- $t \models \exists x \cdot \varphi(x)$ iff there exists $j \in N$ such that $t \models \varphi[j/x]$.

Above $\varphi[j/x]$ denotes the result of replacing every free occurrence of x by j. The notions of satisfiability, validity etc are standard. Note that the following formula is valid:

$$\exists x \bigcirc \varphi(x) \equiv \bigcirc\exists\varphi(x)$$

The following are examples of some typical types that can be specified in the syntax:

- Play action a and b alternatively:

  $$(\ominus p_a@i \supset p_b@i) \wedge (\ominus p_b@i \supset p_a@i)$$

  where p_a and p_b stand for "play action a" and "play action b" respectively.

- Play the action played by the visible player who received the maximum payoff in the previous round:

  $$\exists x(i \leftrightarrow x \wedge r@x \wedge p_a@x) \supset \bigcirc(p_a@i)$$

  where r and p_a stand for "payoff is greater than that of all neighbours of i" and "plays action a (for some $a \in A$)" respectively.

- If there exists a player j within the visibility of i who is in a different neighbourhood X' but plays the same action and gets a better payoff in round k, player i joins the neighbourhood X' of such a player with the maximum such payoff.

  $$\forall x((i \leftrightarrow x \wedge q@x \wedge r@y \wedge \neg[i, x]) \supset \bigcirc[i, x])$$

  where q and r are propositions which say, "payoff is greater than that of i" and "payoff is greater than that of all neighbours of i".

and so on.

Call two formulas $\varphi_1, \varphi_2 \in \Phi$ equivalent, denoted $\varphi_1 \equiv \varphi_2$ if $t \models \varphi_1$ if and only if $t \models \varphi_2$ for all $t \in \mathcal{T}_{\mathcal{C}}(c_0)$. Since π is a fixed map and every neighbourhood graph is finite, we can show the following:

PROPOSITION 1. *Every formula $\varphi \in \Phi$ is equivalent to a quantifier free formula $\varphi' \in \Phi$.*

COROLLARY 1. *The satisfiablity problem for the logic Φ is decidable.*

Remark We remark here that the logic is a standard modal logic on trees, extended to speak of players and neighbourhoods. Our proposal here is not to initate a logical study of neighbourhood switching, but use standard logical machinery to specify a wide variety of rules that constitute the rationale of players for switching strategies or neighbourhoods. The expressiveness of the logic has a critical bearing on game dynamics and hence needs a more careful study. Note that the modalities are branching (as they are interpreted on tree nodes); path connectives like *until* would be meaningful but require a different technical development.

4. STATIONARINESS

In this section we study the dynamics of the games with neighbourhood structures. We are interested in finding out what kind of neighbourhood structures eventually arise and whether the players settle down to playing in such a way that the neighbourhood structure and the actions do not change any further. We look at games where the types of the players are given as formulas. We show that in this case, it is decidable whether the game becomes eventually stationary for the notion of stationariness that we shall define presently.

DEFINITION 1. *A configuration c is said to be stationary if $c' = c$ for all $c \to^* c'$.*

DEFINITION 2. *A game is said to be eventually stationary if it always reaches a stable configuration.*

4.1 Types Specified as Formulas

Let Γ_Φ be a subset of types where every type $\gamma \in \Gamma_\Phi$ is specified as a formula in Φ. Let $\langle \gamma_1, \ldots, \gamma_n \rangle$ be the types of the players where $\gamma_i \in \Gamma_\Phi$ for all $i \in N$. Given neighbourhood graph G and such a type specification, what does it mean for players to play according to their types? Note that the configuration transition relation, $c \to c'$, is derived from player types. Hence, we say that the tree unfolding $\mathcal{T}_{\mathcal{C}}(c_0)$, where c_0 is the initial configuration, conforms to the specification $\langle\gamma_1,\ldots,\gamma_n\rangle$ when $t_0 \models \gamma_1 \wedge \ldots \wedge \gamma_n$ where t_0 is the root of $\mathcal{T}_{\mathcal{C}}(c_0)$.

The implication of such a definition of conformance is as follows: suppose that the types of two or more players are inconsistent. For example, it can be that the types γ_i and γ_j of players i and j are $\bigcirc[i,j]$ and $\neg\bigcirc[i,j]$ respectively. But in that case the formula $\gamma_1 \wedge \ldots \wedge \gamma_n$ is unsatisfiable and hence there is no successor configuration at the node where this formula must hold. This is equivalent to the convention that the game terminates immediately in such situations.

We first study the stationariness of games when the types are specified as formulas from the syntax Φ. That is, for every $i \in N$, $\gamma_i \in \Gamma_\Phi$. In this case we have:

THEOREM 1. *Let $(\mathcal{G}, c_0)$ be an initialised game where $c_0 = (G_0, \mathbf{a}_0)$ and player types $\gamma_1,\ldots,\gamma_n$ specified as formulas in Φ. Then it can be effectively decided whether the game unfolding that conforms to this player types specification eventually becomes stationary.*

PROOF. Let for all $i \in N$, $CL(\gamma_i)$ be the subformula closure of γ_i and $AT(\gamma_i)$ be the set of atoms (propositional consistent sets) of the type γ_i. We construct an atom graph $\mathcal{A} = (V(\mathcal{A}), E(\mathcal{A}))$ as follows:

- $V(\mathcal{A}) \subseteq C \times \Pi_{i\in N} AT(\gamma_i)$ such that
$$(c, \langle D_1,\ldots,D_n\rangle) \in V(\mathcal{A})$$
iff for all $i \in N$, $D_i \cap \mathcal{P} = val_{out}[G](i)$ where $c = (G, \mathbf{a})$.
- A node $w = (c, \langle D_1,\ldots,D_n\rangle) \in V[\mathcal{A}]$ is called *initial* if $c = c_0$ and for all $i \in N$, D_i does not have any formula of the form $\ominus\alpha$. Let $init(\mathcal{A})$ be the set of initial nodes.
- A node $w = (c, \langle D_1,\ldots,D_n\rangle) \in V[\mathcal{A}]$ is called *final* if c is stationary and for all $\Diamond\beta \in D_i, \beta \in D_i$, and for all $\bigcirc\beta \in D_i,\ \beta \in D_i$.
- For $w, w' \in V(\mathcal{A})$ such that
$$w = (c, \langle D_1,\ldots,D_n\rangle)$$
$$w' = (c', \langle D'_1,\ldots,D'_n\rangle)$$
$(w, w') \in E(\mathcal{A})$ iff
  - For all $i \in N$,
    - for all $\ominus\alpha \in CL(\gamma_i)$, if $\alpha \in D_i$ then $\ominus\alpha \in D'_i$,
    - for all $\bigcirc\alpha \in CL(\gamma_i)$, if $\bigcirc\alpha \in D_i$ then $\alpha \in D'_i$,
    - for all $\Diamond\alpha \in CL(\gamma_i)$, if $\Diamond\alpha \in D_i$ then $\alpha \in D'_i$ or $\Diamond\alpha \in D'_i$ and
    - for all $\diamondminus\alpha \in CL(\gamma_i)$, if $\diamondminus\alpha \in D'_i$ then $\alpha \in D_i$ or $\Diamond\alpha \in D_i$.

We call a subgraph $\mathcal{A}'$ of $\mathcal{A}$ *good* if for every node $w = (c, \langle D_1,\ldots,D_n\rangle) \in \mathcal{A}'$:

1. there exists $w' = (c', \langle D'_1,\ldots,D'_n\rangle) \in \mathcal{A}'$ reachable in $\mathcal{A}'$ from w such that w' is final, and
2. there exists $w' = (c', \langle D'_1,\ldots,D'_n\rangle) \in \mathcal{A}'$ such that w is reachable in $\mathcal{A}'$ from w' and w' is initial.

Let $reach(\mathcal{A})$ be the subgraph of $\mathcal{A}$ generated by all the configurations reachable from $init(\mathcal{A})$ in $\mathcal{A}$.

Let $\mathcal{A}'$ be a good subgraph of $\mathcal{A}$ and w_0 be an initial node. Let $\mathcal{T}_{\mathcal{A}}(w_0)$ be the tree unfolding of $\mathcal{A}$ from w_0. $\mathcal{T}_{\mathcal{A}}(w_0)$ is an infinite tree with nodes labelled with elements from $V[\mathcal{A}]$. Let t be a node of $\mathcal{T}_{\mathcal{A}}(w_0)$ such that t is labelled with $(c, \langle D_1,\ldots,D_n\rangle)$. We can show by an easy induction that

CLAIM 1. *For every $i \in N$, for every $\alpha \in CL(\gamma_i)$, $t \models \alpha$ iff $\alpha \in D_i$.*

Thus by the above claim, for a formula $\alpha \in CL(\gamma_i)$ for some $i \in N$, to check if $t \models \alpha$ it is enough to check if $\alpha \in D_i$.

CLAIM 2. *The game is eventually stationary if and only if $reach(\mathcal{A})$ has a good subgraph.*

Let us assume the claim. We see that the construction of the configuration graph $\mathcal{A}$, the reachable subgraph $reach(\mathcal{A})$ and checking whether $reach(\mathcal{A})$ has a good subgraph can all be effectively done. Hence the theorem follows. The claim itself is easy to prove, by observing the connection between good subgraphs and configuration graphs of eventually stationary games. □

5. UNKNOWN TYPES

We have seen above that the restricted expressiveness of types specified by the logic gives us an algorithm for checking stationariness. Can we say anything about the stationariness of games where the types of the players are not known? In general, types may depend on history and hence require unbounded memory. However, we can characterise these games in terms of potentials á la Monderer and Shapley [11]. We show that such games eventually stabilise if and only if they are "well-behaved" in terms of the potentials of configurations.

Since the types of the players are arbitrary, we do not require the logical language to specify them. Hence the utility of the normal form games is given as a function
$$u_a : \mathbf{Y} \to \mathbb{Q}$$
for every $a \in A$. For any neighbourhood X of size k and given a distribution $\mathbf{y}$ of actions of the players in X, $u_a(\mathbf{y})$ gives the utility to all the players in X who play action a.

A game now is a tuple $\mathcal{G} = (typ, \{u_a\}_{a\in A})$ and an initialised game is a pair $(\mathcal{G}, c_0)$ where c_0 is a configuration from the set of configurations C where a configuration as before, is a neighbourhood graph labelled with the actions of the players. The configuration graph $\mathcal{C}$ and the tree unfolding of $\mathcal{C}$ from a configuration $c \in C$ is denoted as $\mathcal{T}_{\mathcal{C}}(c)$ and is defined as before.

5.1 Types of Types

A type γ of a player is said to be finite memory if there exists a finite set M, the memory of the type, $m^I \in M$, the initial memory and functions $\delta : C \times M \to M$, the memory update function and $g : C \times M \to 2^{(2^N \times A)}$, the choice function where for every history $\rho = c_0 \ldots c_k \in H$ if $m_0 \ldots m_{k+1}$ is a sequence determined by $m_0 = m^I$ and $m_{i+1} = \delta(c_i, m_i)$ then $\gamma(\rho) = g(c_k, m_{k+1})$.

A type γ is memoryless if M is a singleton. A memoryless type only depends on the current configuration. That is, if for $\rho, \rho' \in H$ if $last(\rho) = last(\rho')$ then $\gamma(\rho) = \gamma(\rho')$.

5.1.1 Memoryless Types

Theorem 2. *Let $(\mathcal{G}, c_0)$ be an initialised game such that the type of every player is memoryless. $(\mathcal{G}, c_0)$ is eventually stationary if and only if we can associate a potential ϕ_k with every round k such that if the game moves to a different configuration from round k to round $k+1$ then $\phi_{k+1} > \phi_k$ and the maximum possible potential of the game is bounded.*

Proof. One direction is trivial: if such a potential exists, then the tree of possible configurations is finite, stabilising at leaf nodes.

For the other direction, assume that the game eventually stabilises. Then $\mathcal{T}_\mathcal{C}(c_0)$ has the following properties:

1. From the definition of stationariness (definition 2) every branch of $\mathcal{T}_\mathcal{C}(c_0)$ eventually ends in a path such that the associated configuration does not change. That is for every branch b of $\mathcal{T}_\mathcal{C}(c_0)$, there exists a node t at a finite depth such that every successor of t has a unique child and for every successor t' of t, $c(t) = c(t')$. Call t a leaf node and remove the subtree of $\mathcal{T}_\mathcal{C}(c_0)$ rooted at t. After removing such a subtree from every branch of $\mathcal{T}_\mathcal{C}(c_0)$ we get a finite tree $\mathcal{T}_\mathcal{C}^{fin}(c_0) = (T_{fin}, E_{fin})$.

2. Along every branch of $\mathcal{T}_\mathcal{C}^{fin}(c_0)$, for every node t on that branch, there does not exist an ancestor t' of t such that $c(t) = c(t')$. Otherwise, we would have a cycle on the configuration $c(t)$ and since the types are memoryless, this would contradict the assumed eventual stationariness of the game.

We now assign a potential ϕ to every node of $\mathcal{T}_\mathcal{C}^{fin}(c_0)$ such that when ϕ is lifted to the configurations C of the game, ϕ is unique for every configuration $c \in C$. The potential ϕ assigned inductively.

1. For the root node, t_0 say, let $\phi(t_0) = 1$ and let $C_0 = \{t_0\}$.

2a. Suppose C_k has been constructed where C_k is a prefix closed set of nodes of $\mathcal{T}_\mathcal{C}^{fin}(c_0)$. Let $\phi_{\max} = \max\{\phi(t) \mid t \in C_k\}$. Let the boundary of C_k, $B(C_k)$ be the nodes in C_k which have a child outside C_k, i.e., $B(C_k) = \{t \in C_k \mid \exists t' \in T_{fin},\ t \to t',\ t' \notin C_k\}$. Let the interface of C_k, $I(C_k) = \{t \in T_{fin} \mid t' \to t,\ t' \in B(C_k)\}$. Let $clear(C_k) = T_{fin} \setminus \{C_k \cup I(C_k)\}$. We claim that $t \in I(C_k)$ such that there does not exist $t' \in clear(C_k)$ such that $c(t) = c(t')$. Set $\phi(t) = \phi_{\max} + 1$.

2b. For all $t' \in I(C_k)$ such that $c(t) = c(t')$, let $\phi(t') = \phi(t)$. Let $C_{k+1} = C_k \cup \{t\} \cup \{t' \in I(C_k) \mid c(t) = c(t')\}$.

After ϕ has been assigned to all the nodes in $\mathcal{T}_\mathcal{C}^{fin}(c_0)$, we let for every $c \in C$, $\phi(c) = \phi(t)$, $t \in T_{fin}$ such that $c(t) = c$. Note that the potentials have been so assigned that they satisfy the condition $c(t) \to c(t')$ implies $\phi(t) < \phi(t)$. To complete the proof we have to show that step 2a can always be performed.

Suppose, for contradiction, that step 2a cannot be performed for a prefix closed set C_k during the induction. That is, suppose for all $t \in I(C_k)$ there exists $t' \in clear(C_k)$ such that $c(t) = c(t')$.

Now let $t_1 \in I(C_k)$. By our assumption, there exists $t' \in clear(C_k)$ such that $c(t') = c(t_1)$. Let t'_1 be such a node. Let t_2 be the ancestor of t'_1 such that $t_2 \in I(C_k)$. Again by assumption there exists $t'' \in clear(C_k)$ such that $c(t'') = c(t_2)$. Let t'_2 be such a node and let t_3 be the ancestor of t'_2 such that $t_3 \in I(C_k)$. Continuing this way we have a sequence

$$t_1, t'_1, t_2, t'_2, t_3, t'_3, \ldots$$

Now as the tree $\mathcal{T}_\mathcal{C}^{fin}(c_0)$ is finite, it is finitely branching. Hence the above process cannot go on forever and a configuration has to repeat. Suppose t_r be such that the ancestor of t_r in $I(C_k)$ is t_m for some $m < r$. But now since the types of the players are memoryless, this means that $t_m, t_r, t_{r-1}, \ldots, t_m$ forms a cycle of configurations when the players play according to their types. This violates property 2 of the tree unfolding $\mathcal{T}_\mathcal{C}(c_0)$ of the game as mentioned above (see figure 3). □

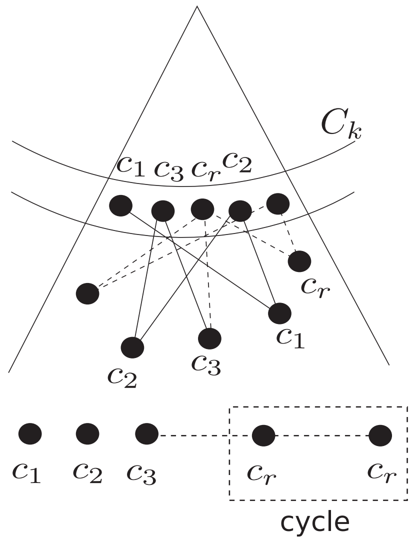

Figure 3: Step 2a of the proof of theorem 2

5.1.2 Stability

The notion of stationariness introduced in the previous section is a bit too rigid. As in the example of the vegetable seller in section 1, a periodic visit to markets A, B and C in that order is an instance of stable behaviour for us. We wish to capture such a behaviour in our notion of stability.

In this subsection we introduce another notion of stable behaviour which we call eventual stability. A cycle C in a graph is called simple if every vertex (except the first) occurs at most once in C.

Definition 3. *A set C of configurations is called stable if C is either a simple cycle with respect to the follows relation $\to$ or a singleton. We say that a game eventually stabilises if it always ends in a stable set of configurations.*

Note that we do not allow complex cycles in a stable set of configurations because then it would make even non-deterministic plays stable. If a complex cycle C consists of two simple cycle C_1 and C_2 then the players can eventually settle down to C even by playing C_1 and C_2 without any regular order. But for a simple cycle C, though the players have a non-deterministic choice of whether to remain in C or to exit C, note that once they exit C they cannot come back to it again. Thus, they are eventually either in a (simple) cyclic play or the configuration of the game doesn't change anymore, both of which are accepted notions of stability for us. But, of course, it is just a matter of choice. One may have other, perfectly justifiable notions of stability.

5.1.3 General Types

In this subsection, we prove a theorem similar to theorem 2 for the case when the types of the players are unknown but arbitrary (not just memoryless).

THEOREM 3. *Let $(\mathcal{G}, c_0)$ be an initialised game. $(\mathcal{G}, c_0)$ eventually stabilises if and only if we can associate a potential ϕ_k with every round k such that the following holds:*

1. *If the game has not yet stabilised in round k then there exists a round $k' > k$ such that $\phi_{k'} > \phi_k$.*
2. *There exists $k_0 \geq 0$ such that for all $k, k' > k_0$, $\phi_k = \phi_{k'}$. That is, the potential of the game becomes constant eventually.*
3. *The maximum potential of the game is bounded.*

PROOF. For the non-trivial direction, assume that the game eventually stabilises. Then from the definition of eventual stability (Definition 3) the tree $\mathcal{T}_{\mathcal{C}}(c_0)$ has the following property. Along every branch b of $\mathcal{T}_{\mathcal{C}}(c_0)$ there exists a node t at a finite depth such that b is just a path from t onwards and either of the following holds:

- b ends in a self-loop: that is, for every successor t' of t, $c(t') = c(t)$. Call t a leaf node and remove the subtree rooted at t.
- b ends in a simple cycle of configurations: that is the following holds. t has a successor t' along b such that $c(t) = c(t')$. Let $t_{\min}$ be the least such successor and let t'' be the parent of $t_{\min}$. Let ρ be the path from t to t'' in $\mathcal{T}_{\mathcal{C}}(c_0)$. It is the case that from t, the branch b is just a sequence of sets of nodes $\rho\rho_1\rho_2\ldots$ such that $|\rho| = |\rho_1| = |\rho_2| = \ldots$ and for every $i \geq 1$ and every $j : 0 \leq j < |\rho|$, $c(\rho_i(j)) = c(\rho(j))$. In other words, the configurations along ρ keep repeating in b forever from t in the same order.

  Call t a leaf node and remove the subtree rooted at t.

After the above procedure we have a finite tree $\mathcal{T}_{\mathcal{C}}^{fin}(c_0) = (T_{fin}, E_{fin})$. We assign a potential ϕ to every node of $\mathcal{T}_{\mathcal{C}}^{fin}(c_0)$ such that when ϕ is lifted back to the configurations C of the game, the requirements of the theorem are satisfied. The potential ϕ is assigned inductively.

Initially let $C_0 = \emptyset$ and $\phi^0_{\max} = 0$. Suppose C_k has been constructed where C_k is a prefix closed set of nodes of $\mathcal{T}_{\mathcal{C}}^{fin}(c_0)$ and let $\phi^k_{\max}$ be the maximum potential of any node in C_k. Let $I(C_k) = \{t \in T_{fin} \mid t \notin C_k, t' \to t, t' \in C_k\}$ be the interface of C_k as in the proof of Theorem 2. We construct a set of *critical* nodes $crit(C_k) \subseteq T_{fin} \setminus C_k$ inductively as follows.

- Let $crit^0(C_k) = \{t\}$ where $t \in I(C_k)$ is an arbitrary node.
- Suppose $crit^i(C_k)$, $i \geq 0$ has been constructed. If there exists $t \in crit^i(C_k)$ be such that there exists $t' \in T_{fin} \setminus (C_k \cup crit^i(C_k))$ with $c(t') = c(t)$ then we construct $crit^{i+1}(C_k)$ as follows. We let $T_t \subseteq T_{fin} \setminus C_k$ be $T_t = \{t' \in T_{fin} \setminus (C_k \cup crit^i(C_k)) \mid c(t') = c(t)\}$.

  Let $cls(T_t)$ be the upward closure of the nodes in T_t till the interface of C_k. That is, $cls(T_t) = \{t' \in T_{fin} \setminus C_k \mid \exists t'' \in T_t, t'$ is an ancestor of $t''\}$. We let $crit^{i+1}(C_k) = crit^i(C_k) \cup cls(T_t)$.

Since $\mathcal{T}_{\mathcal{C}}^{fin}(c_0)$ is finite, there exists a $j \geq 0$ such that $crit^{j+1}(C_k) = crit^j(C_k)$. We set $crit(C_k) = crit^j(C_k)$. Put $\phi(t) = \phi^k_{\max} + 1$ for every $t \in crit(C_k)$ and set $C_{k+1} = C_k \cup crit(C_k)$. Note that C_{k+1} is a prefix-closed set and for every node $t \in C_{k+1}$, there does not exist $t' \in T_{fin} \setminus C_{k+1}$, such that $c(t') = c(t)$ (otherwise t' would have been added to $crit(C_k)$ while processing t).

After ϕ has been assigned to all the nodes in $\mathcal{T}_{\mathcal{C}}^{fin}(c_0)$, we let for every $c \in C$, $\phi(c) = \phi(t)$, $t \in T_{fin}$ such that $c(t) = c$. Note that the process of saturation in the construction of the critical sets ensures that if a node t is assigned a potential at some iteration then all nodes t' such that $c(t') = c(t)$ are assigned the same potential in the same iteration. This guarantees the uniqueness of the potential for every configuration. Also the assignment of the potentials in a top-down fashion on the unfolding $\mathcal{T}_{\mathcal{C}}^{fin}(c_0)$ of the configuration graph, makes sure that the other requirements of the theorem are satisfied. □

5.1.4 Finite Memory Types

From the proof of theorem 3 we easily have the following theorem:

THEOREM 4. *Let $(\mathcal{G}, c_0)$ be an initialised game. If $(\mathcal{G}, c_0)$ eventually stabilises then the types of all the players are finite memory.*

PROOF. Assume that $(\mathcal{G}, c_0)$ stabilises. Let $\mathcal{T}_{\mathcal{C}}^{fin}(c_0)$ be the finite tree as constructed in the proof of theorem 3. Then the required finite memory type of each player i is γ_i such that the memory of γ_i is the set of nodes T_{fin} of $\mathcal{T}_{\mathcal{C}}^{fin}(c_0)$. The initial memory is the root of $\mathcal{T}_{\mathcal{C}}^{fin}(c_0)$. The memory update function is given by the edge relation in $\mathcal{T}_{\mathcal{C}}^{fin}(c_0)$ and the choice function g_i at a memory node $t = (G, c)$ is given by $g_i(c, t) = \{(X, a) \mid (X, a)$ corresponds to the choice of i at a child t' of $t\}$. □

6. WEIGHTED CO-ORDINATION GAMES: AN APPLICATION

To gain intuition into the dynamics of these games with neighbourhood structures, in this section, we study a special case of such games which we call weighted co-ordination games under the assumption that all the players play simple imitative strategies.

First, we formally define these games. For simplicity we assume that the action set of every player is binary, that is $A = \{0, 1\}$. The payoff of the players are determined by the amount of co-ordination in the neighbourhood they belong to. More precisely, let X be a neighbourhood and let $X_0[G_k]$ be the set of players who play the action 0 in round k and $X_1[G_k]$ be the set of players who play the action 1. Then the payoff to a player i in X who plays 0 is given as

$$u_0(|X_0[G_k]|, |X_1[G_k]|) = \frac{|X_0|}{|X|}$$

and the payoff of a player j in X who plays 1 is given as

$$u_1(|X_0[G_k]|, |X_1[G_k]|) = \frac{|X_1|}{|X|} = 1 - u_0(|X_0[G_k]|, |X_1[G_k]|)$$

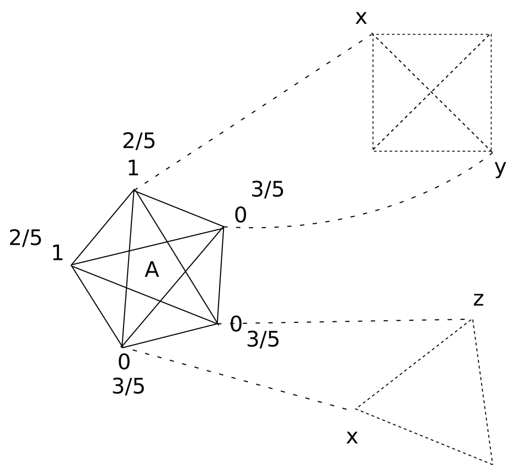

Figure 4: An example of a weighted co-ordination game. The players playing 1 in the neighbourhood A receive a payoff of 2/5 whereas those playing 0 receive 3/5.

See Figure 4 for an example.

We show that when all the players are of a simple imitative type (to be defined presently), we can associate a potential to every configuration c which is bounded and such that if c' follows c then its potential is strictly greater than c. Hence by theorem 2, such games always stabilise. We can also give an upper bound on the number of rounds required to attain stability.

We first describe a simple imitative type t. We assume that every player is of type t. Suppose player i plays a in round k where $a \in \{0, 1\}$. If in round k player i receives a payoff less than 0.5 and there exists a player j visible to her who in round k received the maximum payoff among all the players visible to her, then in round $k+1$ i plays the action of j. This type t is given by the following formula α_s:

$$\forall x.(p@i \wedge i \leftrightarrow x \wedge q@x \wedge r@x \wedge p_a@x) \supset \bigcirc(p_a@i)$$

where p, q, r and p_a are propositions which say:

- p: payoff is less than 0.5.
- q: payoff is greater than that of i.
- r: payoff is greater than that of all neighbours of i.
- p_a: plays action a (for some $a \in A$).

If it is a dynamic neighbourhood game, the type is the following: if there exists a player j within the visibility of i who is in a different neighbourhood X' but plays the same action and gets a better payoff in round k, player i joins the neighbourhood X' of such a player with the maximum such payoff. This is given by a very similar formula α_d:

$$\forall x.(p@i \wedge (i \leftrightarrow x \wedge q@x \wedge r@x \wedge \neg[i, x]) \supset \bigcirc[i, x]$$

Theorem 5. *Let $\mathcal{G}$ be a game with initial neighbourhood graph G and let all the players be of the same type t defined by α_s or α_d, depending on whether the neighbourhood structure is static or dynamic, respectively. Then:*

1. *In the static neighbourhood case, let m be the number of neighbourhoods (cliques) and $M = \max_{X \in clq(G)} |X|$. Then the game always stabilises and it does so in at most mM steps.*
2. *In the dynamic neighbourhood case, the game always stabilises in at most $n^{n(n+1)/2}$ steps.*

Proof outline The proof is by assigning a potential to every configuration in such a way that 2 can be used to infer eventual stability.

In the static neighbourhood case, at any round k define the potential of a neighbourhood X to be $\phi_k(X) = \max\{|X_0|, |X_1|\}$. Then define the potential of the game in round k to be $\phi_k = \sum_X \phi_k(X)$.

In the dynamic neighbourhood case, observe that possible payoffs of a player in any round are $1/n, 2/n, \ldots, 1/(n-1), 2/(n-1), \ldots, 1$. We arrange them in ascending order. We now define the potential of a payoff p in round k inductively as follows: $\phi_k(1/n) = 1$, $\phi_k(p') = \phi_k(p) + 1$ where p is the immediate predecessor of p' in the ordering of the payoffs. These potentials can be lifted to the vertices (players) as $\phi_k(i) = \phi_k(u_{a(i)}(X(i)))$. We define the potential of a neighbourhood X in round k as $\phi_k(X) = \sum_{i \in X} \phi_k(i)$. Finally we define the potential of the game in round k as $\phi_k = \sum_X \phi_k(X)$ where the sum is over the set of all cliques in the neighbourhood graph G_k for round k.[2] □

Remark When there is an upper bound on the size of any clique (say M) in a dynamic neighbourhood structure, then the number of steps to stability can be bounded in terms of M. In the worst case, this makes no difference, but may be significant in some applications.

7. DISCUSSION AND FUTURE WORK

What we have presented here is merely a preliminary discussion on neighbourhood structures in games and a satisfactory theory has some way to go. First, mixed strategies and expectations of other players are more interesting in such a framework. A player would then strategise based on the expected payoffs of other players and we would talk about the probabilities of players switching between different neighbourhoods. Secondly, the logic we have introduced for specifying the types of players is quite limited. What would be more interesting is a logic with the power of quantification over various cliques of the neighbourhood graph. Since the cliques of the graph keep changing, such a logic would be highly expressive and may lead to better logical foundation for this study. A better approach would be to study the logical structure of player types *per se* from an expressiveness perspective and limit game dynamics accordingly.

Another important question is what the structures of the stable configurations are; viz. whether it is a collection of a large number of small neighbourhoods or is it a few large neighbourhoods. Such questions are interesting for economists as well.

Acknowledgement

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[2]The full proof can be found in the extended version at `www.imsc.res.in/~soumya/Files/Neighbourhood.pdf`

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